THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 26, 1902, extending through the usual morning and afternoon sessions. Over forty persons were in attendance, including the following thirty-seven members of the Society :

Professor E. W. Brown, Dr. J. E. Clarke, Professor F. N. Cole, Mr. J. L. Coolidge, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Mr. Peter Field, Dr. William Findlay, Professor T. S. Fiske, Miss Carrie Hammerslough, Professor James Harkness, Professor L. I. Hewes, Dr. E. V. Huntington, Professor Harold Jacoby, Dr. S. A. Joffe, Dr. Edward Kasner, Dr. C. J. Keyser, Dr. G. H. Ling, Professor W. H. Metzler, Mr. H. B. Mitchell, Professor E. H. Moore, Dr. I. E. Rabinovitch, Professor J. K. Rees, Professor E. D. Roe, Miss I. M. Schottenfels, Professor I. J. Schwatt, Professor Charlotte A. Scott, Mr. Ferdinand Shack, Professor D. E. Smith, Dr. Virgil Snyder, Dr. H. F. Stecker, Dr. W. M. Strong, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor L. A. Wait, Miss E. C. Williams, Dr. Ruth G. Wood.

The President of the Society, Professor Eliakim Hastings Moore, occupied the chair, yielding it during the afternoon session to Professor T. S. Fiske. The Council announced the election of the following persons to membership in the Society : Professor C. E. Biklé, Columbia University ; Professor F. W. Duke, Hollins Institute, Va. ; Dr. J. G. Hardy, Williams College ; Professor H. L. Hodgkins, Columbian University ; Dr. J. N. Ivey, Tulane University ; Dr. J. H. McDonald, University of California ; Dr. H. C. Moreno, Stanford University ; Dr. T. M. Putnam, University of California ; Dr. E. W. Rettger, University of California ; Mr. W. H. Roever, Harvard University ; Professor Irving Stringham, University of California ; Dr. S. D. Townley, University of California ; Mr. H. E. Webb, Stevens School, Hoboken, N. J. ; Mr. A. W. Whitney, University of California. Three applications for admission to membership were received.

The Council also authorized the organization of a Pacific Section of the Society, to hold meetings in the vicinity of San Francisco. The first meeting of the new Section was held May 3. A report of this meeting will appear in the next issue of the BULLETIN.

The following papers were read at the April meeting:

(1) Dr. H. F. STECKER: "The curve of least contour in the non-euclidean plane."

(2) Mr. J. L. COOLIDGE: "Quadric surfaces in hyperbolic space."

(3) Dr. F. H. SAFFORD: "Dupin's cyclides of the third degree."

(4) Mr. PETER FIELD: "On the forms of plane unicursal quintic curves."

(5) Dr. RUTH G. WOOD: "Non-euclidean displacements and symmetry transformations."

(6) Mr. D. R. CURTISS: "Note on the sufficient conditions for an analytic function."

(7) Miss I. M. SCHOTTENFELS: "On the definitional functional properties for the analytical functions

$$\frac{\sin \pi z}{\pi}, \frac{\cos \pi z}{\pi}, \frac{\tan \pi z}{\pi}."$$

(8) Professor C. A. SCOTT: "On the circuits of plane curves."

(9) Dr. E. V. HUNTINGTON: "A complete set of postulates for the theory of real numbers."

(10) Dr. L. P. EISENHART: "Surfaces whose lines of curvature in one system are represented on the sphere by great circles."

(11) Professor E. H. MOORE: "A definition of abstract groups."

(12) Dr. E. V. HUNTINGTON: "A second definition of a group."

(13) Mr. L. D. AMES: "Evaluation of slowly convergent series."

(14) Dr. EDWARD KASNER: "Groups of Cremona transformations and systems of forms."

(15) Mr. A. D. RISTEEN: "The constant of space."

(16) Dr. C. J. KEYSER: "Concerning the angles and the angular determination of planes in 4-space."

(17) Professor T. J. I'A. BROMWICH: "The infinitesimal generators of parameter groups."

(18) Professor T. J. I'A. BROMWICH: "On the parabolas, or paraboloids, through the points common to two given conics or quadrics."

Mr. Ames's paper was communicated to the Society through Professor Bôcher, Professor Bromwich's papers through the Secretary; Mr. Risteen was introduced by Professor P. F. Smith. In the absence of the authors, the papers of Dr. Safford, Dr. Keyser, and Professor Bromwich were read by title and those of Mr. Curtiss and Mr. Ames were read by Dr. Huntington. The papers of Mr. Curtiss and Dr. Keyser appeared in the May number of the BULLETIN; those of Professor Bromwich and the second paper of Dr. Huntington are contained in the present number. Abstracts of the other papers are given below.

Dr. Stecker's paper applies the methods of the calculus of variations to the question of the curve of least contour in the non-euclidean plane, and establishes the fact that the curve sought is such that its non-euclidean distance from a fixed point is constant. It also follows that such a non-euclidean circle and the curve in the non-euclidean plane having constant Gaussian curvature are identical.

The metrical properties of a surface in hyperbolic space depend upon its relation to the absolute. A classification of the curves common to quadric surfaces and the absolute is given by Mr. Coolidge by means of Weierstrass's "elementary divisors." The various forms of surface are then taken up in detail and examined with regard to circular sections, focal conics, equations, and general appearance.

Dr. Safford's paper discusses cyclides derived from confocal cones of the second order and gives a method of generation by means of the diametral planes. The equations, in rectangular coördinates, of the three orthogonal families are considered, together with necessary conditions for the elimination of irrelevant conclusions as to intersections and lines of curvature.

The only papers devoted to the forms of plane unicursal quintic curves are two by Meyer and one by Dowling. In the first paper by Meyer, the method of knots is applied to curves having six real distinct crunodes, the forms of the curves are given, but no reference is made to equations or parametric representation. In Meyer's second paper the same results are obtained by quadric inversion. In the paper by Dowling one paragraph is devoted to the general equation of a unicursal quintic. No figures are given. Mr. Field completes the classification of unicursal quintics by a new method, giving the form and equation of each type. The definition of a type is essentially that proposed by Meyer. In the case of six distinct double points, a nodal cubic and a conic are used for basic curves. In case of a triple point, a line and a double conic have the same office; and finally for a fourfold point the equation $u_4 + u_5 = 0$ is used. The positions of the real inflections are given for the last set.

The problem considered in Dr. Wood's paper has been suggested by Study in the *Leipziger Berichte* for 1890, and the results obtained for non-euclidean displacements and symmetry transformations are similar to those obtained by him for euclidean space. The equations of a non-euclidean line reflection (or skew reflection on conjugate polars of the quadric) are first found by means of the definition: A non-euclidean line reflection is the product of inversion in two complexes of the pencil $\alpha_i + \lambda\beta_i$, $\beta_i + \lambda\alpha_i$ $(i = 1, 2, 3)$ which are in involution. These equations are of the form

$$y_1' = \lambda(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \beta_1^2 - \beta_2^2 - \beta_3^2)y_1 + 2(\alpha_2\beta_3 - \alpha_3\beta_2)y_2 + 2(\alpha_3\beta_1 - \alpha_1\beta_3)y_3 + 2(\alpha_1\beta_2 - \alpha_2\beta_1)y_4,$$

and similarly for y_2', y_3', y_4', where $\sum_{i=1}^{3}\alpha_i\beta_i = 0$.

These equations can be found immediately from Cayley's formulæ for the homographic transformation of the quadric surface into itself if we set $\alpha_0 = \beta_0 = 0$. The law of composition of parameters of two such transformations given by Cayley furnishes an analytic proof of this theorem that every non-euclidean displacement (*i. e.*, a collineation of space which leaves the two systems of generators of the quadric surface invariant) may be compounded of two non-euclidean line reflections, and show that the directrices of the two line reflections belong to the complex which serves in defining the resulting displacement. The other two of Cayley's parameters α_0 and β_0 are found to be proportional to the non-euclidean distance and the cotangent of the non-euclidean distance between the directrices of the two line reflections of which the displacement is compounded. The equations for a non-euclidean symmetry transformation, *i. e.*, the collineations of space which interchange the two systems of generators of the quadric surface) are found by means of the theorem that such a transformation is compounded of a reflection upon an arbitrary plane as $x_1 = 0$ and a non-euclidean line reflection. These equations are the same as Cayley's "improper" transformation. The invariant planes, points, and the axis of rotation of such a transformation are readily found.

Miss Schottenfels's paper deals with the determination of the simplest set of definitional functional properties for the analytic functions

$$f(z) = \frac{\sin \pi z}{\pi},\ \frac{\cos \pi z}{\pi},\ \frac{\tan \pi z}{\pi}.$$

It is known that there exists a quartic with two nodes, having the property of being met by every straight line in at least two real points. The general theorem enunciated and proved in Professor Scott's paper is that for every order n there exist curves ($p = 0$ or 1), composed of a single circuit, met by every straight line in at least $n - 2$ real points; moreover, there exist curves composed of a single circuit, met by every straight line in at least $n - 4$, $n - 6$, $n - 8$, $\cdots$, real points. The proof is obtained by means of Cremona transformations. The nature of the curves is most clearly seen when they are projected on a sphere by lines through the center.

Dr. Huntington's paper on the postulates of all real numbers (positive, negative and zero) is a continuation of his paper on the postulates of magnitude (positive real number), published in the *Transactions* for April, 1902. In the present paper those assemblages are considered in which 1° a rule of combination $a \circ b$, and 2° a sub assemblage Q are so defined as to satisfy the following eight postulates:

1. If a and b are elements of the assemblage, there is an element x such that $a \circ x = b$.

2. If a, b, and either $a \circ b$ or $b \circ a$ belong to the assemblage, then $a \circ b = b \circ a$.

3. If a, b, c, $a \circ b$, $b \circ c$, and either $(a \circ b) \circ c$ or $a \circ (b \circ c)$ belong to the assemblage, then $(a \circ b) \circ c = a \circ (b \circ c)$.

4. If a and b belong to the sub-assemblage Q, then $a \circ b$, if it belongs to the assemblage at all, will also belong to Q.

5. If there is a peculiar element e such that $c \circ e = c$ for every element c, then e does not belong to Q.

6. If there is a peculiar element e such that $c \circ e = c$ for every element c, and if $a \circ b = e$ while $a \neq e$, then either a or b belongs to Q.

7. (Let the notation $a < b$ indicate that an element x of Q exists, such that $a \circ x = b$.) If S is an infinite sequence (a_k) of elements of Q, such that $a_k < a_{k+1}$ and $a_k < c$ ($k = 1, 2, 3, \cdots$), where c is some element of Q, then there is an element A having the following two properties: 1° $a_k < A$ whenever a_k belongs to S; 2° if $A' < A$ then there is at least one element of S, say a_r, for which $A' < a_r$.

8. If a is an element of Q, there are two elements of Q, say x and y, such that $x \circ y = a$.

Every assemblage which satisfies these eight postulates is shown to be equivalent to the system of real numbers, and the independence of the postulates from one another is established.

Dr. Eisenhart's paper is in abstract as follows: Guichard has shown that the determination of all congruences, whose developables have a given spherical representation, reduces to the solution of an equation of Laplace, after the direction cosines of the lines have been found. We apply this method to the case where one family of lines on the sphere is composed of great circles and the other consists of their orthogonal trajectories. It is evident that all of these congruences are normal and that the lines on the sphere are the images of the lines of curvature of the parallel surfaces which cut the lines orthogonally. This furnishes a means for the determination and study of all surfaces whose lines of curvature in one system are represented on the sphere by great circles.

In the first place, it is found that, when such a configuration is given upon the sphere, the determination of the direction cosines of the lines of the congruence is the same problem as the finding of a skew curve from its intrinsic equations. For a given family of great circles there is a double infinity of families of surfaces whose lines of curvature in one system have this representation. Surfaces of this kind are characterized by the property that one family of lines of curvature are geodesics, and along the lines of curvature of the second system one of the principal radii is constant; and, furthermore, one of the sheets of the evolute is a developable surface and conversely. Hence these surfaces are the moulded surfaces of Dupin, and consequently can be generated by a plane curve whose plane rolls without sliding upon a developable surface. In the integration of the equation of Laplace two arbitrary functions are introduced; one of these determines the generating curve and the other joins with the spherical representation to fix the developable generator. The second sheet of the evolute is a moulded surface with the same surface generator and the generating curve is the evolute of the curve of the surface; this sheet and the involutes are represented on the sphere by the same great circles and corresponding points are at the distance of a quadrant.

Surfaces of revolution are moulded surfaces, corresponding to the case where the surface generator is a straight line. The complete determination of all the surfaces of revolution whose meridians are represented by a possible family of great circles reduces to quadratures. Surfaces of revolution are the only moulded surfaces which are Weingarten surfaces and they are the only isothermic moulded surfaces.

The definition of abstract groups given by Professor Moore is an adaptation, in the light of that given by Dr. Huntington in the April (1902) BULLETIN, of a type of definition introduced independently some years ago by Professor Pierpont and himself. A set of elements $a, b, \cdots$, with a multiplication table, constitutes a group if: 1°. Every product $a \circ b$ belongs to the set. 2°. For every three elements a, b, c such that $a \circ b$, $b \circ c$, $a \circ (b \circ c)$, and $(a \circ b) \circ c$ belong to the set, $a \circ (b \circ c) = (a \circ b) \circ c$. 3_l°. An element i_l exists such that for every element a, $i_l \circ a = a$. 3_r°. An element i_r exists such that for every element a, $a \circ i_r = a$. 4_l°. If an element i_r (of character 3_r) exists, for some such element i_r for every element a an element a_l' exists such that $a_l' \circ a = i_r$. 5°. (a) The number of elements is a positive integer n, or (b) the set contains an infinitude of elements. The postulates 1, 2, 3_l, 3_r, 4_l, $5a$, and also the postulates 1, 2, 3_l, 3_r, 4_l, $5b$ are then shown to be independent of one another.

The following is an abstract of Mr. Ames's paper: Any convergent series

$$u_1 + u_2 + u_3 + \cdots \tag{1}$$

can be written

$$\frac{1}{2} u_1 + \frac{1}{2^2}(u_1 + u_2) + \frac{1}{2^3}(u_1 + 2u_2 + u_3) + \cdots, \tag{2}$$

or $\frac{1}{2} u_1 - \frac{1}{2^2} \triangle^1 |u_1| + \frac{1}{2^3} \triangle^2 |u_1| + \cdots$, where $\triangle^k |u_n|$ is the nth term of the kth order of differences of $|u_1| + |u_2| + \cdots$. This converges and equals (1). In many cases it converges rapidly when (1) converges slowly or diverges. Let R_k be the remainder in (2) after k terms. If (1) is alternating and $\triangle^k |u_n|$ and $\triangle^{k+1} |u_n|$ have opposite signs which remain unchanged for $n \geqq 1$, then $|R_k| < \left| \frac{1}{2^k} \triangle^k |u_n| \right|$. A sufficient condition for this is that $f(x)$ exists such that $f(n) = |u_n|$ and $\frac{d^k}{dx^k} f(x)$ and $\frac{d^{k+1}}{dx^{k+1}} f(x)$ have opposite signs which remain unchanged for $x > 1$. If the hypergeometric series $F(\alpha, \beta, \gamma, x)$ is convergent and $x < 0$, it always satifies the first condition if a suitable number of terms be dropped at the beginning. This number can be determined in any particular case by computing the general term of the particular order of differences desired. The sum of the first K terms of (2) is $\frac{1}{2^k}[\lambda_1 u_1 + \lambda_2 u_2 + \cdots \lambda_n u_n]$ where $\lambda_r = 1 + k + \frac{k(k-1)}{1\ 2} + \cdots$ to $k - r + 1$ terms. In numerical computation it is usually better to compute the successive orders of differences. Simple inspection will in many cases make it highly probable without further tests that the degree of accuracy reached is much greater than that indicated above. Successive repetition of the process by which (2) was obtained from (1) gives

$$\begin{gathered}\frac{1}{p+1}u_0 + \frac{1}{(p+1)^2}(pu_0 + u_1) \\ + \frac{1}{(p+1)^3}(p^2u_0 + 2pu_1 + u_2) + \cdots,\end{gathered} \tag{3}$$

where p is arbitrary. Even when (1) is divergent, p can in many cases be so chosen that (3) converges. If (1) is a power series, and (3) converges throughout a larger region than (1), it yields the analytic continuation of the function represented by (1) throughout this larger region.

The groups considered in Dr. Kasner's paper are related to the various complete systems of invariants considered in the theory of forms. Let $f_1, f_2, \cdots, f_n$ be a set of forms of the same dimension and order, and $I_1, I_2, \cdots, I_\nu$ a system of invariants which is complete in the sense that any invariant of the forms is a rational function of the members of the system. Then if $f_{a_1}, f_{a_2}, \cdots, f_{a_n}$ denote any permutation of the forms, and $I_1^{(a)}, I_2^{(a)}, \cdots, I_\nu^{(a)}$ the corresponding system of invariants, this system is rationally expressible in terms of the original system, and the passage from the latter to the former constitutes a Cremona transformation. The transformations thus induced by all possible permutations are proved to form a group. The group is isomorphic to the symmetric permutation group of n letters, the isomorphism being in general simple, though in special cases it may be multiple.

All the groups obtained from the same set of forms by this process are equivalent within the infinite group consisting of all Cremona transformations. A distinct group is obtained, however, by means of the complete system of absolute invariants. The binary linear forms give rise in this way to the cross ratio group discussed by Professor Moore in the *American Journal of Mathematics*, 1900. The groups discussed in detail are those relating to any set (n) of linear forms in any number (k) of variables for both the associated system and the system of absolute invariants, the number of variables being

$$\binom{n}{k} - \binom{n-k}{k} \quad \text{and} \quad (k-1)(n-k-1)$$

respectively. The process considered is capable of extension in several directions, for example to forms which are not all of the same order, and to the formation of infinite groups.

Mr. Risteen's paper is in abstract as follows: If Euclid's parallel axiom is true, then it is known that the parallax of a double star can be obtained in either of two ways, namely, 1° by the usual micrometric measures, and 2° by observing the relative velocities of approach or recession of the component stars in the line of sight. If the parallel axiom is not true, and space is admitted to be hyperbolic, then the known trigonometric relations that hold for hyperbolic space, and which were given by Lobachevsky, enable us to combine the micrometric measures and the spectroscopic ones, so as to obtain a single estimate of the star's distance, together with the value of the "constant of space" that occurs in Lobachevsky's equations, but whose value has not yet been determined. The same principle applies equally well if space is elliptic. Lobachevsky's method of finding a limiting value for his constant is unsound. A numerical value of the "constant of space" can hardly be found at present, because the necessary spectroscopic data cannot yet be had.

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THE INFINITESIMAL GENERATORS OF PARAMETER GROUPS.

BY PROFESSOR T. J. I'A. BROMWICH.

(Read before the American Mathematical Society, April 26, 1902.)

§1. DR. SLOCUM has given (BULLETIN, January, 1902, page 156) a method for calculating the infinitesimal gene-