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SOME INSTRUCTIVE EXAMPLES IN THE CALCULUS OF VARIATIONS.

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IN the following note I propose to give some examples which illustrate in a simple manner several points of fundamental importance in the calculus of variations.

§ 1. *The General Problem and its Assumptions.*

We consider the problem * to minimize the integral

$$I = \int_{x_0}^{x_1} F(x, y, y')dx \tag{1}$$

under the following *assumptions:*

1. The function $F(x, y, z)$ considered as a function of the three independent variables x, y, z is real and regular † in the vicinity of every finite real point $x = a$, $y = b$, $z = c$ for which $x = a$, $y = b$ lies in a given region R of the xy-plane.

2. The functions $y = f(x)$ admitted to consideration satisfy the following conditions:

(a) For the given end values $x = x_0$, $x = x_1$, y takes the given values $y = y_0$ and $y = y_1$ respectively, *i. e.*, the "curves" $y = f(x)$ pass through two given points A: (x_0, y_0) and B: (x_1, y_1);

* Compare for the formulation of the problem Osgood's article "Sufficient conditions in the calculus of variations," *Annals of Math.*, 2nd ser., vol. 2 (1901), p. 105. We deviate, however, in several points from Osgood's assumptions.

† *i. e.* developable into an ordinary power series in $x - a$, $y - b$, $z - c$.

(*b*) y is continuous on the whole interval (x_0x_1), and the first derivative y' exists and is continuous on (x_0x_1) with the possible exception of a finite number of points; in these exceptional points the progressive and regressive derivatives exist and are finite;

(*c*) The "curves" $y = f(x)$ lie, for $x_0 \leqq x \leqq x_1$ in the interior of the region R.

The totality of functions (curves) satisfying these conditions will be denoted by $\mathfrak{M}$.

A function $y = f(x)$ of $\mathfrak{M}$ is said to *minimize* the integral I if there exists a positive quantity ϵ such that the total variation

$$\Delta I = \int_{x_0}^{x_1} F(x, \bar{y}, \bar{y}')dx - \int_{x_0}^{x_1} F(x, y, y')dx$$

is positive for every function $\bar{y} = \bar{f}(x)$ of $\mathfrak{M}$, different from y, for which

$$|\Delta y| < \epsilon,$$

where

$$\Delta y = \bar{y} - y.$$

It is well known that the following conditions are *necessary* for a minimum:

1. The minimizing function $y = f(x)$ must be an *extremal*, *i. e.*, satisfy *Euler's differential equation:* *

$$F_y - \frac{d}{dx} F_{y'} = 0, \tag{I}$$

literal subscripts denoting partial derivatives.

2. *Legendre's condition:*

$$F_{y'y'}\left(x, f(x), f'(x)\right) \geqq 0 \text{ on } (x_0x_1). \tag{II}$$

3. *Jacobi's condition:* †

$$x_1 \leqq x_0' \tag{III}$$

where x_0' denotes the "conjugate" of x_0, x_0 being supposed less than x_1.

* See for instance Whittemore, *Annals of Math.*, 2nd ser., vol. 2 (1901), p. 130.

† See Erdmann, *Schlömilch's Zeitschrift*, vol. 23 (1878), p. 367.

4. *Weierstrass's condition:* *

$$\frac{E\big(x, f(x), f'(x), p\big)}{\big(p - f'(x)\big)^2} \geqq 0 \tag{IV}$$

on $(x_0 x_1)$ and for every finite value of p, the function

$$E(x, y, y', p)$$

being defined by

$$\begin{aligned} E(x, y, y', p) = F(x, y, p) - F(x, y, y') \\ - (p - y') F_{y'}(x, y, y'). \end{aligned} \tag{2}$$

Since by Taylor's theorem

$$E(x, y, y', p) = \frac{(p - y')^2}{2} F_{y'y'}\big(x, y, y' + \theta(p - y')\big) \tag{3}$$

where $0 < \theta < 1$, it follows that (IV) is always satisfied whenever

$$F_{y'y'}\big(x, f(x), p\big) \geqq 0 \tag{II$_a$}$$

on $(x_0 x_1)$ for every finite value of p.

By (II′), (II$_a$′), (III′), (IV′), we denote the inequalites (II), (II$_a$), (III), (IV) after the suppression of the equality sign.

§ 2. *Example Illustrating the Necessity of Weierstrass's Condition.*

After Jacobi had discovered condition (III), it was generally believed that conditions (I), (II′), (III′) were also *sufficient* for a minimum, until Weierstrass proved in 1879 their insufficiency and added his fourth necessary condition.

The following example illustrates this point:

Example I: To minimize the integral

$$I = \int_{x_0}^{x_1} y'^2 (y' + 1)^2 dx. \tag{4}$$

* Weierstrass, Lectures, 1879; compare also Osgood, *l. c.*, p. 118, and Hilbert, *Archiv für Mathematik* (3), vol. 1 (1901), p. 231. The expansion of E according to powers of $p - y'$ shows that the quotient $E(x, y, y', p) / (p - y')^2$ remains regular even when $p = y'$.

Here the extremals are straight lines; let

$$\mathfrak{C}: \qquad y = mx + n$$

be the extremal through the two given points $A: (x_0, y_0)$ and $B: (x_1, y_1)$.

Further, we have

$$F_{y'y'} = 2(6y'^2 + 6y' + 1).$$

Hence

$$F_{y'y'}\big(x, f(x), f'(x)\big) = 2(6m^2 + 6m + 1).$$

The roots of the equation

$$6m^2 + 6m + 1 = 0$$

are

$$m_1 = \tfrac{1}{2}(-1 + 1/\sqrt{3}) = -0.2113 \cdots$$

$$m_2 = \tfrac{1}{2}(-1 - 1/\sqrt{3}) = -0.7887 \cdots$$

Hence, if we suppose the two given points A and B so situated that

$$\text{either} \quad m > m_1 \quad \text{or} \quad m < m_2,$$

condition (II′) will be satisfied.

Further, condition (III′) is always satisfied; for since the extremals through the point A are straight lines, it follows that there exists no conjugate point on $\mathfrak{C}$, however far we may go.

Finally

$$\frac{E\big(x, f(x), f'(x), p\big)}{\big(p - f'(x)\big)^2} = p^2 + 2(m+1)p + 3m^2 + 4m + 1;$$

it is positive for every p when $m(m+1) > 0$; it can change sign when $m(m+1) < 0$.

Accordingly we have to distinguish the following three cases:

Case I: $-1 < m < 0$.

In this case *the line AB does not minimize the integral I.* This can be seen at once as follows: However small we may select a positive quantity ϵ, we can always draw in the "neighborhood (ϵ)" * of AB a broken line $\bar{\mathfrak{C}}$ joining A and B and

* Compare, for the definition of neighborhood, Osgood, *l. c.*, p. 106.

made up of segments having alternately the slope 0 and -1. For such a broken line the integral $\bar{I}$ is evidently equal to

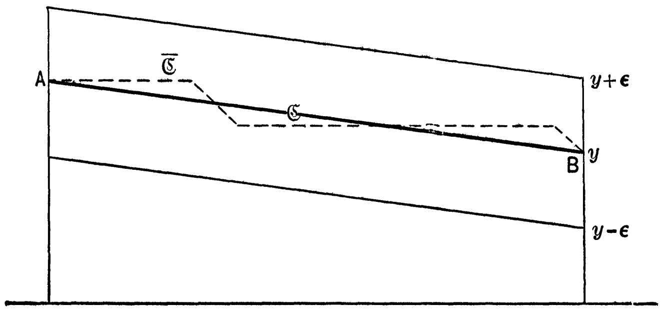

FIG. 1.

zero, the integrand being zero between the limits of integration, whereas for the line AB the integral I has the positive value

$$I_{\mathfrak{C}} = m^2(m+1)^2 \cdot (x_1 - x_0).$$

Case II: $m = 0$ or $m = -1$.

In this case the *line AB minimizes the integral I;* for it furnishes the value $I = 0$ whereas every other curve joining A and B and representable in the form $y = f(x)$ furnishes a positive value.

Case III: $m > 0$ or $m < -1$.

In this case the *line AB minimizes the integral I.* For the set of straight lines parallel to AB constitutes a "field of extremals" about AB, hence by Weierstrass's * theorem we have for every curve $\overline{\mathfrak{C}}$ not coinciding with AB, joining A and B and satisfying the conditions of § 1,

$$\Delta I = \int_{x_0}^{x_1} E(x, y, y', \bar{y}')dx, \tag{5}$$

where x, y is a point of $\overline{\mathfrak{C}}$; $\bar{y}'$ the slope of $\overline{\mathfrak{C}}$ at x, y; y' the slope at x, y of the extremal of the field passing through x, y, *i. e.*, $y' = m$; therefore

$$E(x, y, y', \bar{y}') = (\bar{y}' - m)^2[\bar{y}'^2 + 2(m+1)\bar{y}' + 3m^2 + 4m + 1],$$

* Compare Osgood, *l. c.*, p. 118; also Hilbert, *Archiv für Mathematik* (3) vol. 1 (1901), p. 231.

which is positive except where $\bar{y}' = m$. Hence it follows that

$$\Delta I > 0.$$

Our example illustrates in this case another point, viz., that *condition* (II_a) *is not necessary for a minimum.* Indeed

$$F_{y'y'}\big(x, f(x), p\big) = 2(6p^2 + 6p + 1),$$

which may take negative as well as positive values.

§ 3. *Relation between a Problem in Parameter Representation and the Corresponding Problem with x as Independent Variable.*

When one has become acquainted with Weierstrass's treatment of the simplest type of problems of the calculus of variations (viz., in parameter representation) one is inclined to regard the older method — in which x is taken as the independent variable — as antiquated and imperfect as compared with Weierstrass's method; unjustly however, for *the two methods deal with two clearly distinct questions* and which of the two deserves the preference depends upon the nature of the special problem under consideration.

When it is proposed to minimize the integral

$$I = \int_{t_0}^{t_1} F\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}\right) dt, \tag{6}$$

where

$$F(x, y, kx', ky') = k\, F(x, y, x', y') \tag{7}$$

for every positive k, we consider a certain well-defined manifoldness $\mathfrak{N}$ of *curves*

$$x = \phi(t), \quad y = \psi(t),$$

out of which we have to select those curves which minimize the integral I.

If we consider, instead of $\mathfrak{N}$, another manifoldness $\mathfrak{M}$ of curves, we obtain an entirely different problem.

Let now $\mathfrak{M}$ be, in particular, the totality of those curves of $\mathfrak{N}$ which are representable in the form

$$y = f(x),$$

$f(x)$ being a single valued function of x. Then the new problem may be formulated thus: Among all curves (*functions*) $y=f(x)$ of the totality $\mathfrak{M}$ to find those which minimize the integral

$$I=\int_{x_0}^{x_1} F\left(x, y, 1, \frac{dy}{dx}\right) dx. \tag{8}$$

We distinguish the two problems as "the problem (T)" and "the problem (X)." Since $\mathfrak{M}$ is contained in $\mathfrak{N}$, it follows that *every solution of (T) which is representable in the form $y=f(x)$ is a fortiori also a solution of (X); but (T) may have solutions which are not representable in this form and which accordingly are not solutions of (X).*

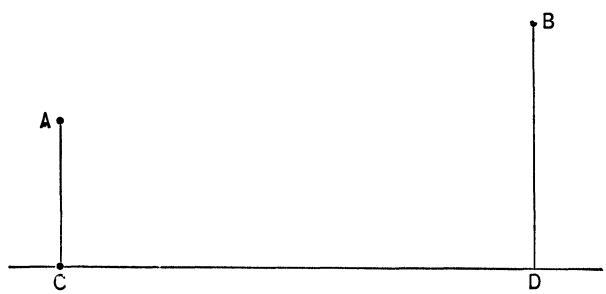

FIG. 2.

A well-known example of this kind is the so-called "discontinuous solution" of the problem of the minimum surface of revolution,* which consists of two perpendiculars AC and DB to the axis of revolution (the x-axis) and the segment CD of the latter. The broken line $ACDB$ is under certain assumptions concerning the positions of A and B a solution of the problem (T), but not of the problem (X), because the segments AC and BD are not representable in the form $y=f(x)$.

On the other hand, *the problem (X) may have solutions which are not at the same time solutions of (T).*

An example of this kind is furnished by the problem of § 2, case III. In this case the line AB furnishes, as we have seen, a smaller value for the integral

* See Kneser, Lehrbuch der Variationsrechnung, p. 179.

$$I=\int_{x_0}^{x_1} y'^2 \left(+ y' 1\right)^2 dx,$$

than any other curve joining A and B, and representable in the form $y=f(x)$. But if we pass to the corresponding problem (T) and admit also curves which are not representable in this form, we can easily find neighboring curves which furnish still smaller values for the integral I than the line AB; for instance any broken line $\bar{\mathfrak{C}}$ joining A and B and made up of

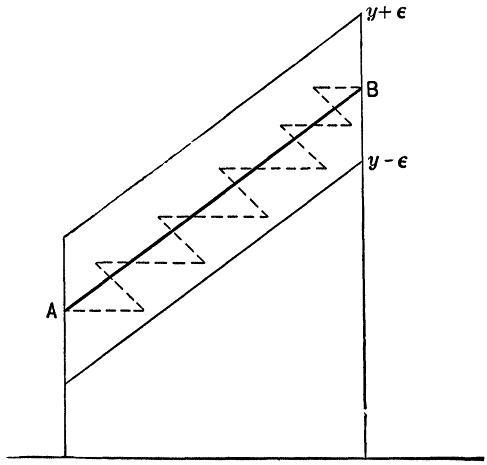

FIG. 3.

segments having alternately the slope 0 and -1. For such a broken line the integral is zero, whereas it is positive for the line AB. Hence AB is a solution of (X) but not of (T).

§ 4. *Example Illustrating the Insufficiency of Weierstrass's Condition in Case x is Taken as Independent Variable.*

Weierstrass discusses the problem to minimize the integral (6) under the assumption that $F(x, y, x', y')$, considered as a function of its four arguments is regular in the vicinity of every point $x=a$, $y=b$, $x'=a'$, $y'=b'$ for which a, b lies in a certain region of the xy-plane, while a', b' may have any system of values except $a'=0$, $b'=0$. Under this assumption he proves that the conditions corresponding to (I), (II′), (III′), (IV′) are *sufficient* for a minimum.

Nevertheless, for a minimum of the integral (1) under the assumptions stated in § 1, *the conditions* (I), (II'), (III'), *and* (IV') *are not sufficient, not even if* (IV') *be replaced by the stronger condition* (II'_a).*

This is shown by the following example:

Example II.—To minimize the integral

$$I = \int_0^1 (ay'^2 - 4byy'^3 + 2bxy'^4)dx, \tag{9}$$

a, b being two positive constants, with the initial conditions

$$y = 0 \text{ for } x = 0 \quad \text{and} \quad y = 0 \text{ for } x = 1.$$

Here Euler's equation reduces to

$$- y'' F_{y'y'} = 0, \tag{I}$$

where
$$F_{y'y'} \equiv 2a - 24byy' + 24bxy'^2.$$

The only extremal through the two given points $A : (0, 0)$ and $B : (1, 0)$ is the straight line

$$\mathfrak{C}: \qquad y = 0.$$

Further

$$F_{y'y'}\big(x, f(x), f'(x)\big) \equiv 2a > 0. \tag{II$'$}$$

The set of extremals through A is the pencil of straight lines through A ; hence there exists no conjugate point and condition (III$'$) is satisfied.

Further

$$\frac{E(x, y, y', p)}{(p - y')^2} \equiv (a - 8byy' + 6bxy'^2)$$
$$+ p(-4by + 4bxy') + 2bxp^2.$$

Hence along $\mathfrak{C}$

$$\frac{E\big(x, f(x), f'(x), p\big)}{\big(p - f'(x)\big)^2} \equiv a + 2bxp^2 > 0. \tag{IV$'$}$$

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* This statement seems to contradict directly the theorem on sufficient conditions given on page 118 of Osgood's article. The contradiction is, however, only apparent since Osgood makes the further assumption—see page 108 —that "$F_{y'y'}(x, y, p)$ does not vanish in R" from which it follows that if $F_{y'y'}(x, y, p)$ is positive at all points of $\mathfrak{C}$ for every value of p, it is also *positive at all points* OF A CERTAIN NEIGHBORHOOD OF $\mathfrak{C}$, *for every value of* p.

The four conditions (I), (II′), (III′), (IV′) are therefore satisfied; even the stronger condition

$$F_{y'y'}\big(x, f(x), p\big) \equiv 2a + 24bxp^2 > 0. \qquad (\mathrm{II}'_a)$$

Nevertheless the line $\mathfrak{C}$ *does not minimize the integral I.*

For if we replace the line AB by the broken line APB, the

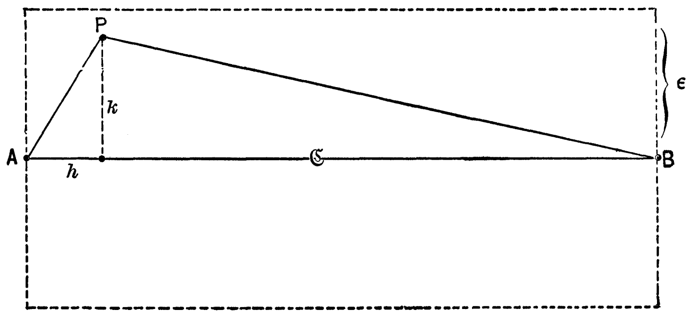

FIG. 4.

coördinates of P being $x = h > 0$ and $y = k$, the total variation of I is easily found to be

$$\Delta I = k^2\left[-\frac{bk^2}{h^2} + \frac{a}{h} + a + 3bk^2\right] + (h) \qquad (10)$$

where
$$\underset{h=0}{\mathrm{L}}(h) = 0.$$

Now let $\epsilon > 0$ be given as small as we please, then choose $|k| < \epsilon$ and let h approach zero, keeping k fixed. Then since $b > 0$ it follows that

$$\Delta I < 0$$

for all sufficiently small values of h, which proves that the line AB does not minimize the integral I.

The explanation of the fact that the four conditions in question are not sufficient in the one problem whereas the corresponding four conditions are sufficient in the other problem is as follows: In Weierstrass's problem no restriction is imposed upon the direction of the tangents of admissible curves, whereas in our problem the direction parallel to the y-axis, and no other direction, is excluded. In the former case the totality of admissible directions is closed, in the latter it is not closed.

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