the effects of planetary and other disturbing influences, are much to be commended except the paragraph in the center of the last page. In this the astonishing statement is made that, because the moon's orbit shrinks as the eccentricity of the earth's orbit decreases, the moon would fall on the earth if the earth's orbit became circular. Obviously a variable may always decrease without having zero as its limit, and in this case the incorrectness of the conclusion is quite evident.

In recapitulation it may be said that Professor Brown has written a very satisfactory treatise on the lunar theory, the faults being for the most part those which have their origin in the inherent, and up to the present unsolved, difficulties of the subject. There is no better source of information for one who wishes to acquaint himself with this celebrated field of investigation.

F. R. MOULTON.

THE UNIVERSITY OF CHICAGO,
November, 1902.

THE DOCTRINE OF INFINITY.

Die Grundsätze und das Wesen des Unendlichen in der Mathematik und Philosophie. Von Dr. K. GEISSLER. Leipzig, Teubner, 1902. 4to, 417 pp.

THE recent tendency toward critical investigation of the axioms underlying mathematical sciences has led to the necessity of discussing the philosophical basis of the whole mathematical structure, and it is to be hoped that more attention will be given the subject from a purely philosophical standpoint.

The announcement of a book treating the old stumbling block of the infinite from both the mathematical and the philosophical points of view was therefore of the greatest interest, and the character of the book was apparently guaranteed by the excellent reputation of the publishers. But the book — though not lacking in a certain kind of interest — is in several respects most surprising, and, as we shall see, somewhat disappointing.

The first 296 pages are devoted to mathematical investigations, and we shall consider these alone. A great variety of topics are treated, such as the parallel axiom, indeterminate forms, tangents, limits, derivates, velocity, curvature and many other problems involving continuity. It is not here proposed to discuss these subjects in detail, but to confine ourselves to one serious matter, which the author regards as fundamental and which certainly affects the whole of his reasoning. This is his introduction of certain (fixed) infinitely great and infinitely small numbers (or segments in geometry).

There are two reasons for considering the subject seriously. In the first place it is common, in certain elementary treatises on the calculus, to introduce such infinitely small quantities and to deal with them under the name of "differentials." From the standpoint of mathematical pedagogy, it is certainly of vital importance to discover to what conclusions the introduction of such numbers leads, both with regard to the correctness of using them at all and with regard to the effect upon the student's conceptions and his mode of thought. Secondly, from the point of view of the foundations of mathematics, it is interesting to investigate the axioms tacitly assumed and to see just what axioms (of the usual set) may be retained and what new axioms need be added in the logical treatment of such infinitely great and infinitely small quantities, capable of assuming fixed constant values.

One of the axioms generally accepted in both arithmetic and geometry is that of Archimedes, which may be stated for arithmetic as follows: *Given any two positive numbers a and b, of which b is the greater, there exists a certain positive integer n, such that* $a + a + a + \cdots (n \text{ times}) > b$.

Passing over the first few pages of the book, which are intended for very elementary students, we find, on page 35, in direct violation of Archimedes's axiom, the introduction of a certain positive ($\neq 0$) number δ, termed an "infinitely small" number, such that

$$\delta + \delta + \delta + \cdots (n \text{ times}) < 1$$

for any (finite) integer n. And the "fundamental theorem" is repeatedly stated (*e. g.*, page 141) as follows: "The product of a finite quantity and an infinitesimal is always infinitesimal"; while on page 66 it is stated that "the concept of an infinitesimal does not coincide exactly with zero." *

These statements explicitly deny the axiom of Archimedes, though the author does not mention the fact. Nor will the

* See also p. 46; and p. 90, where an infinitesmal unit is considered.

non-euclidean geometer protest against a geometry in which certain usual axioms are expressly violated, provided the fact of such violation is constantly kept in mind and no appeal whatever is made to ordinary geometry. In particular, one axiom which is at the basis of all considerations of continuity in the ordinary sense having been rejected, we naturally expect a thorough research of the axioms of continuity which are assumed to hold. We shall see that the author has unfortunately not adhered to such a programme.

No result of the book can then be accepted, even for a non-euclidean, non-archimedean, geometry. But it is interesting to note that some of the results obtained are such as actually do hold in the non-archimedean geometry actually constructed by Hilbert, in his Grundlagen der Geometrie.

If we consider as our realm of numbers all the rational functions of a certain parameter t, and define "equality," "greater than," and "less than" by the rule $f(t) >, =, < g(t)$ according as $f(t) - g(t) >, =, < 0$ for all sufficiently large positive values of t, then all the axioms of arithmetic (see Hilbert, page 26), except the axiom of Archimedes, are satisfied.*

We find, for instance, in the book in hand, on page 62, that "a ratio of two infinite [or infinitesimal] numbers of the same class is a finite number, * * * and a product of two infinite numbers of the same class is not finite, but is infinite [or infinitesimal] of another class." Arrangement into classes might be made by the rule that numbers of any one class satisfy, among themselves, the axiom of Archimedes. It is then readily seen that the above statement holds true in Hilbert's non-archimedean geometry. The two numbers

$$\frac{a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0}{b_n t^n + b_{n-1} t^{n-1} + \cdots + b_0} \text{ and } \frac{c_r t^r + c_{r-1} t^{r-1} + \cdots + c_0}{d_s t^s + d_{s-1} t^{s-1} + \cdots + d_0}$$

are of the same class if $m - n = r - s$. Their quotient is of the class $(m - n) - (r - s) = 0$, and is hence "finite," *i. e.*, in the same class with 1; while their product is of class $(m - n) + (r - s)$ which is neither zero nor equal to $(m - n)$ unless $m - n = 0$. This latter exceptional case corresponds to the statement (page 62): "The class of finite numbers is that in which the product and the ratio of any two numbers of the class again belongs to the class" — which holds for positive

* The form is slightly different from Hilbert's.

numbers. We are also informed that the choice of which class is to be considered finite is not at our disposal.

On page 177 the author says "the motion of sliding" — (presumably continuous motion in euclidean space) "is to be thought of not in general, but only for a certain class of segments at a time." The notion expressed practically amounts to saying that the complete (non-archimedean) geometry assumed does not permit of continuous motion in the ordinary sense. The author is at great pains elsewhere * to explain away the difficulty that a passage from "finite" to "infinite" [or to "infinitesimal"] numbers, for instance, can be affected *neither* by a sudden spring, nor yet *without* such a spring. That he has violated a fundamental axiom of continuity, and hence cannot expect his geometry to be continuous in the ordinary sense, is not made clear.

Again, on page 194, for example, the author is surprised to find himself in difficulties with such words as "continuous," "hole-less," etc., as applied to space. And he declares that a segment (in geometry) or a number (in arithmetic), when added to a (greater) second, "does not alter the value of the second except when it is of the same class." And on the next page (195): "for this reason and only for this reason we shall be correct if we consider in any discussion the numbers (or segments) of any one class." But this is returning to archimedean geometry, and relinquishing our previous violation of the archimedean axiom. And, unfortunately for this assertion, numbers and segments of different classes *are* treated in the same discussion constantly throughout the book (see pages 28, 29, 51, 59, 77, etc.).

Finally we find, on page 196, "the limit of a (finite) variable x is a region (precisely expressed, a region of finiteness); and this region is infinitely small, but otherwise arbitrary."

Such are, indeed, in part, the necessary conclusions of a non-archimedean geometry. But it should be carefully noted that such results have nothing to do with euclidean geometry, *i. e.*, with space as *ordinarily* conceived and treated. It is the author's lack of appreciation of this fact, and his constant appeal to the intuition (and to even much less strict metaphysical persuasion) which robs the book of any value whatever, even as a discussion of non-archimedean geometry. Even the archimedean axiom itself seems to be used implicitly at

* See *e. g.*, p. 10.

points, in assuming that the geometry is continuous in the ordinary sense, but the arguments are so intermingled with intuitional conclusions that it is wholly impossible to decipher just what axioms the author is implicitly assuming. It is scarcely necessary to state that the discussions of limits, infinite series, the parallel axiom,* derivates, etc., are riddled with errors; and the minor matters treated are no less free from gross mistakes. That Leibniz's ideas of infinity are the nearest approach to the ideas of this book, as the author frequently asserts, is in itself a sufficient condemnation.

The moral of the book concerns the thoughtless introduction of "differentials" in our ordinary books on elementary calculus.† A "differential," treated as an "infinitesimal," or "infinitely small quantity," is precisely a non-archimedean number; and no reasoning can be applied to a system of numbers in which such "infinitesimals" occur without an investigation of the extraordinary properties of such a non-archimedean (non-euclidean) arithmetic (or geometry).

On the other hand, if differentials are to be introduced in the calculus, we may do so (in two dimensions) by considering the perfectly finite quantities

$$dx \equiv \Delta x, \quad dy \equiv \frac{dy}{dx} \cdot \Delta x, \quad d_2 y \equiv \frac{d^2 y}{dx^2} \cdot \overline{\Delta x}^2,$$

etc., where Δx is a certain fixed finite increment of x chosen at pleasure. Such a treatment can be made perfectly rigorous, and the advantages (if any such exist) of the differential notation can be preserved, while our geometry and arithmetic remain euclidean and archimedean. But the differentials here considered are perfectly finite quantities, with no taint of peculiarity or obscurity.

The errors in conceptions and in modes of thought, induced in a student by the use of differentials in a loose sense, are well illustrated by the nature of the conceptions of this author. And on serious consideration the treatment of calculus by the use of "infinitely small" differentials must be discarded ‡ by

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* A "proof" of this axiom is given on p. 26. The author states that previous proofs are lacking in rigor!

† And to a minor extent, the ordinary calculus treatment of indeterminate forms, and of infinite series.

‡ This statement of course presupposes that the elementary course on the calculus is to be based on euclidean geometry, and is to use intuitive proofs of theorems involving continuity and limits, to some extent.

any sincere writer as totally illogical in itself, and baneful in its effects upon the student.

While this is but one phase of the book in hand, it is the most important phase, to which all else is made subordinate. We will then note but one other point: namely, that the author protests against a violation of the axiom that two points always determine a straight line (page 44), but assumes that "a whole is greater than a part" even when the "whole" in question is infinite (page 19). In both particulars the modern tendency is certainly quite the opposite.

The philosophical portion of the book will not be criticised here, but it is scarcely felt that a philosopher would care to be sponsor for the statements made — certainly not for the style in which they are presented. The philosophical views of mathematical thought, and in particular of infinity and infinitesimals, must surely take into account, however, the positive results now in the possession of mathematicians regarding the effect of the violation of the archimedean axiom upon our system of axioms and upon our conceptions of space.

E. R. HEDRICK.

SHEFFIELD SCIENTIFIC SCHOOL,
November, 1902.

SOME RECENT GERMAN TEXT-BOOKS IN GEOMETRY.

Lehrbuch der Elementar-Geometrie. Erster Theil, Planimetrie. Von Dr. E. GLINZER. Siebente Auflage. Dresden, Kühtmann, 1899. 4to, 120 pp.

Grundriss der Geometrie. I. Planimetrie. Von J. H. KÜHL. Zweite Auflage. Dresden, Kühtmann, 1900. 4to, 95 pp.

Lehrbuch der Stereometrie. Von Dr. P. SAUERBECK. Stuttgart, Kröner, 1900. 4to, 291 pp.

Die Elemente der analytischen Geometrie. Zweiter Theil: Die analytische Geometrie des Raumes. Von Dr. F. RUDIO. Dritte, verbesserte Auflage. Leipzig, Teubner, 1901. 4to, vi + 186 pp.

THE authors of the first two texts are connected with the institution in Hamburg known as the Allegemeine Gewerbeschule und Baugewerkschule. One might expect accordingly