multipliers which are not unimodular, it can be expressed in the form

$$\phi' = e^{\Sigma c_i w_i}\phi,$$

in which ϕ is a form with unimodular multipliers and the w_i are normal integrals of the first kind on the Riemann surface F. A generator of Γ which corresponds to a crossing of a canonical period path in F will then reproduce ϕ' with a multiplier $e^{c_i p_i}\mu_i$, p_i and μ_i being the corresponding moduli of periodicity of w_i and the multiplier of ϕ respectively. By a proper choice of the c_i it is evident that any desired multiplier system may be assigned to ϕ'.

In order that the form ϕ be expressible as a Poincaré series it is further necessary that it vanish in the parabolic fixed points. After a somewhat lengthy, but interesting analysis, a conclusion is reached similar to that in case $p = 0$, namely — every unimultiplicative form of dimension d satisfying the convergence condition, which vanishes in the parabolic points, and only such a form, is expressible as a Poincaré series; and every automorphic form with arbitrary poles, none of which occur at parabolic points, is expressible as a Poincaré series plus an integral automorphic form.

<div style="text-align: right">J. I. HUTCHINSON.</div>

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LORIA'S SPECIAL PLANE CURVES.

Spezielle algebraische und transcendente ebene Curven, Theorie und Geschichte. Von Dr. GINO LORIA. Autorisierte, nach dem italienischen Manuscript bearbeitete deutsche Ausgabe von FRITZ SCHÜTTE. Leipzig, B. G. Teubner, 1902. 8vo., xxi + 744 pp. + 17 plates containing 174 figures.

ABOUT ten years ago the Royal Academy of Sciences at Madrid offered its triennial prize to be awarded upon the thirty-first of December, 1894, for "An ordered account of all curves of any kind which had received special names, and a further short account of their form, their equations, and their inventors." To this prodigious question no response seems to have come. Three years later the question was repeated. Professor Loria presented his researches, which were received

in a most flattering manner, as well they might be. These re-
searches, in a somewhat altered form, carefully worked out and
translated into German, form the contents of the large volume
here under review.

Although the work is encyclopædic and all-inclusive so far
as it goes, it does not cover the entire question set by the
Academy at Madrid. Those curves, which are made up of
arcs of other curves, and which therefore, though often of great
use in applied mathematics or in the theory of the fine arts, are
not representable by a single analytic equation, have purposely
been omitted. No reference has been made to curves in space.
The investigation of these must be left for a separate volume,
which it is to be hoped Professor Loria will have time and in-
clination to write. In all other respects the author has covered
the immense field of special curves whether named or not
named. Thus, while in one way he has failed to compass the
extent indicated in the announcement of the prize question, in
another way he has exceeded the limits.

As the investigations on special curves often long antedate
the *Jahrbuch* and lie hidden in doctors' theses, in essays, ad-
dresses, letters of a private correspondence, and in larger
treatises where they are introduced only for some immediate
special purpose, there can be little doubt — as the author him-
self modestly states — that some of the curves have not been
attributed to their earliest inventors and that others may have
been overlooked entirely. To find most of the special curves is
not easy ; to find them all is impossible. But probably no one
person will detect more than a few corrections or additions to
this work of Professor Loria, who has made such a profound
and long-continued study of the field. Essential improvements
must be the result of years and of many collaborators.

The method of treatment pursued by the author is, with the
rarest exceptions, analytic. In the case of some curves, like the
cissoid and conchoid, which date back to antiquity, a limited
amount of geometric reasoning has been employed. For the
most part, however, even these oldest curves have been handled
analytically by the methods of rectangular or polar coördinates,
or both. This method of treatment is to be commended, for
only in this manner could be given anything approaching such
a complete and systematic treatment as would interest modern
readers. The large number of foot-notes, referring to the
places where the original treatment of the curves may be

found, will be of invaluable service to those who delight in antiquarian methods or research. Most readers will find in the volume before us enough to satisfy them, except where the author has preferred to refer to some modern work in which a special curve is fully treated, rather than to give that treatment in detail. For example, the cardioid possesses numerous properties, of which only a few are given in the text, leaving the others to be studied in the elaborate thesis of Dr. R. C. Archibald.

Professor Loria's book will appeal in different degrees to a great variety of readers. That everything in the work should be interesting to a large number is inconceivable. Despite the careful and logical arrangement one can not fail to perceive that the problems are distinct and subject to distinct methods of treatment. The theory of special curves must necessarily be a collection of special theories, each applicable in general only to its particular curve. To this statement there are a few exceptions. If the curve happens to be unicursal, that is, expressible rationally in terms of a parameter, there are general theorems and methods which apply. Usually, however, one finds that he has at his disposal no other mathematical implements than elementary analytic geometry and the first principles of the differential and integral calculus. There must be some special motive or no one would read the book through as he might read a work on a more connected field of mathematics. On the other hand there is much here which will interest every one, especially one who teaches the elements of analytic geometry or of calculus. Every teacher knows the value of choosing problems which have interested mathematicians from early times down to the present day. Such problems are often easier for students to handle than the artificial ones invented purposely for exercises. The early mathematicians were not refined experts in analysis, having no great theories on which to draw. So they were forced to invent simple curves with simple properties immediately connected with simple geometric constructions, and these they treated by elementary means. The result is that this work done early in the development of mathematical science furnishes problems valuable in mathematical teaching.

This material has been collected by Professor Loria and put into a form easily accessible to all. For example the chapter on the cissoid of Diocles contains, in addition to the deduction of the equation of the curve, the following properties which,

taken collectively, form a series of interesting problems easy to
prove by elementary methods. " The pedal of the parabola
with respect to the vertex is a cissoid. The locus of points
symmetric to the vertex of a parabola with respect to a (mov-
ing) tangent is a cissoid. If one parabola rolls externally upon
an equal parabola the locus of the vertex is a cissoid. The polar
reciprocal of a cissoid with respect to a circle about the cusp as
center is a semi-cubical parabola. The envelope of the common
chords of a cissoid and its (moving) circle of curvature is a simi-
lar cissoid of which the ordinates for corresponding abscissas are
$\frac{1}{3} \cdot (\frac{7}{3})^{\frac{3}{2}}$ as great. The area between the cissoid and its asymp-
tote is three times the area of the generating circle (Fermat;
geometric interpretation by Huygens communicated to Wallis
in a letter). The center of gravity of this area divides the
distance between the cusp and the asymptote in the ratio $5 : 1$.
The volume generated by the revolution of the cissoid about
its asymptote is equal to that of the ring generated by the same
revolution of the generating circle (Sluse). The difference be-
tween the lengths of the cissoid and its asymptote when the
ordinates y and $- y$ become infinite, approaches a finite quantity
which may be evaluated by means of logarithms."

Here are eight theorems — four in analytic geometry, four in
elementary calculus. These theorems are connected with such
names as Diocles, Fermat, Huygens, Wallis, and Sluse ; and the
footnotes refer to places where these and further theorems may
be found. Similar treatment is given to other curves including
the cardiod, hypocycloid of three cusps, conchoid of Nicomedes,
catenary, and tractrix. The student might well make the ac-
quaintance of some of these famous curves early in his career.
They are certainly more interesting than purely artificial loci.
Then too there are curves in Professor Loria's book which are
discussed by means of hyperbolic and elliptic functions and
offer good exercise in their manipulation.

Most of the important theorems are printed in heavy faced
type. They stand out on the text with an emphasis more than
italic and may easily be found by rapidly turning the pages.
A valuable and pleasing feature of the work is the elaborate
plots of the important curves. There are one hundred and fifty
numbered figures ; but, as many of the numbers are subdivided,
there are actually about one hundred and seventy-five different
plots. The volume ends with an index of names, an index of
subjects, and a special index for the plates. This careful atten-

tion to the form in which the volume is offered to the public is especially welcome in this case as it greatly facilitates rapid reference to any particular curve.

We now pass to a more detailed examination of the work. The first section of three chapters and thirteen pages, deals with the straight line, circle, and conic sections. Naturally the treatment is cursory, being in fact merely a few historical notes upon the most important ideas such as the geometry of a ruler, of a parallel ruler, or of the compasses. The author then passes immediately to higher plane curves.

The second section contains fourteen chapters reaching to the ninety-fourth page of the text and treating of cubics in point coördinates. The classification into five types (Chapter I) and the general treatment of unicursal (rational) cubics (Chapter II) are the most straightforward, neat and elementary that we know. The author, relying on the apology that his treatment is encyclopedic, gives F. W. Newman's classification of cubics into forty-two subtypes, each bearing a botanical or architectural name. It seems as if in this case a reference to the original memoir would be sufficient. In its place might have been given an account of that omitted but interesting work of Möbius which treats curves by means of their projection on a sphere. The circular curves, the cissoid with its generalization, the cubical parabola and folium of Descartes, the oblique and right strophoid, Sluse's conchoid, Rolle's curve, and the various witches, fill up the main part of the section. The two concluding, like the two beginning chapters, are the most interesting. They treat the cubic curves which serve for the trisection of an angle and the duplication of a cube, thus solving in a manner those two celebrated problems of antiquity, of which at least one is supposed to be of Pythian origin.

At this point we enter the only two serious complaints to be made against Professor Loria's work. First, there is practically no attention paid to line-coördinates, homogeneous or non-homogeneous. To be sure in earler times these coördinates were unknown. It is only within the last century or half-century that they have become of importance. Hence there must be very few special curves defined by means of them. Yet in any extended treatise, even on special curves, these coördinates merit more attention than the author gives them — as they also do in elementary instruction where, they can serve as the most simple and important examples of envelopes. We

see in Professor Loria's work numerous cases of isolated or acnodal points; but isolated tangents seem to be lacking. This geometric phenomenon is however more worthy of note as it seems more peculiar and more confusing than in the case of points. This first objection may be only half-hearted, but not so our second.

A more important objection lies in the fact that no mention is made in the text or indexes of Abel or his celebrated theorem which yields beautifully and powerfully some of the most interesting properties of higher plane curves. That the theorem applies in general is no hindrance to its use on special curves. The lack of it is not particularly felt when dealing with rational curves of the third and fourth orders. The author obtains the congruences, which would be given by Abel's theorem, by another means — that of the rational representation. But in any work on algebraic curves, special or otherwise, the total omission of this theorem, which is perhaps the greatest single theorem in the subject, seems scarcely conceivable.

The third section extends to page 218 and contains sixteen chapters. To the first three chapters, on classification of quartics, unicursal quartics, and elliptic and bicircular quartics, the remarks made upon the first two chapters of the preceding section apply. The absence of Abel's theorem begins to be felt. The succeeding chapters treat of Perseus's spiral, Nicomedes's conchoid, three-cusped curves of the fourth order and in particular the hypocycloid, cartesian ovals, polyzomals of the fourth order, rational curves with a node at which the tangents to the two branches are coincident, conchals, cassinian ovals, curves with three double points at each of which both branches possess inflections, the mussel-line, a quartic for trisecting angles, quartics which may be obtained in simple manners from conics. The mussel-line, although not especially interesting for its geometric properties, deserves a passing note because among a number of other things it was invented by Dürer (1525) for use in the fine arts and was said by him " to serve many a useful purpose."

The titles of the chapters of the first three sections have been quoted so fully merely to indicate the vast amount of detail, the great care and exhaustiveness with which the volume before us has been compiled. There appear a number of names doubtless quite unknown to most readers. In many cases the curves themselves possess no very great interest and

are passed by with but short notice.   Some of the chapters are
less than two pages in length; others extend over more than
a dozen.   In the following sections only the more important or
better known curves will be mentioned.

Section IV deals with special curves of order greater than
four.   The orders reach nine and even twenty-five in the case
of some curves connected with lemniscate functions.   Probably
the most interesting curve is Watt's, derived from Watt's par-
allelogram.   The other names such as astroids, scarabeans,
nephroids, atriphtaloids probably represent so little to the
mind as to be not worth quoting.

To this point the curves have so been defined as to possess
a definite order.   The author next passes to curves which may
have any desired order.   Among these are the parabolas and
hyperbolas of order n, the pearl curves, Cayley's general poly-
zomals defined by $\Sigma_i \sqrt{U_i} = 0$, ovals, curves for dividing an
angle in a given ratio, self-polar curves, and curves obtained
by inversion (anallagmatic).   There are at the end of the sec-
tion chapters, which possess a greater interest, upon algebraic
curves whose rectification depends on assigned functions.   The
author gives applications to curves of which the length may be
expressed in terms of arcs of an ellipse or lemniscate.   Taken
in its entirety this fifth section is one of the longest in the book ;
it contains seventeen chapters and over one hundred and fifty
pages.

The sixth section is the longest, with its twenty-five chapters
and almost two hundred pages, and the subject, special trans-
cendental curves, is one of the most interesting.   Among others
are found curves for squaring the circle, spirals of various spe-
cies, the whole trochoid family whether the base upon which
the circle rolls be straight line or circle, the tractrix, catenary,
cross-ratio or W-curves of Klein and Lie, lines of Mercator and
Sumner, curves assumed by one-dimensional elastic bodies, the
herpolhode, and some other curves of mathematical physics.
Here, too, is a short account of the crinkly or otherwise amor-
phic curves.   Strangely enough there seems to be no treatment
of the Γ-curve in the text and the name does not occur in the
index.   The omission appears extraordinary as that curve not
only possesses interesting properties of itself, but aids in the
representation of many functions among which may be cited
Euler's integrals of the second kind.   The whole section is a
veritable treasury of problems of a purely geometric nature

solvable by the use of the simpler transcendental functions, and as such it ought to appeal to those teachers of calculus who prefer not to draw very largely on physics for their problems. There are also excellent examples in the use of intrinsic coördinates and an appendix is devoted to showing how to pass from an equation in intrinsic to the corresponding equation in cartesian coördinates. The intrinsic equations are certainly interesting and important, and it is to be hoped that they will find greater recognition now that Professor Cesàro's work has appeared in German under the title Natürliche Geometrie.*

The seventh and concluding section, in twelve chapters and about one hundred and twenty pages, treats the very general question of curves which may be obtained by certain laws from a given curve. The generalized tractrix, evolutes, involutes, glissettes, parallel and radial curves, caustics, pedals and antipedals, and finally the derivative and integral curves derived by $y = f'(x)$ and $y = \int f(x)dx$ where $f(x)$ itself is defined by an equation $F(x, y) = 0$, form some of the topics. The last is worthy of especial mention owing to the fact that in general the derivative and integral curves are discussed only in the special case where the function $f(x)$ is a polynomial in x.

Although we have reached the end of the book proper, a note of a dozen pages under the unpromising title: Review of the historical development of the theory of plane curves, contains for advanced students a subject of more interest and stimulus than the rest of the volume taken together. This subject is the author's recent work on a class of curves which he calls *panalgebraic* and which are in general transcendental. The last seven pages of the note form a summary of Professor Loria's memoir, originally published in the *Prager Berichte*, and more recently reprinted with additions in the second volume of *Le Matematiche Pure ed Applicate*. Owing to the restricted American circulation of these periodicals the researches on panalgebraic curves are probably not well known among us, and we may be permitted in closing this review to go into detail upon the note with which the author closes his book.

Very truly he says: "What we need is a theory which will include most if not all of the known transcendental curves." Such is his theory of panalgebraic curves: for there are only about a half-dozen known curves which fail to fall under this

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* See Dr. Snyder's review in the BULLETIN for April, 1903, pp. 349–357.

classification.   The author considers a function $F(x, y, y')$ algebraic in each of the three variables x, y, y'.   Without loss of generality we may assume that the function F is rational and integral and that the polynomials in x and y, which serve as coefficients of the powers of y', have no common factor $\phi(x, y)$. Write

$$F(x,\ y,\ y') \equiv \sum_{r=0}^{r=n} f_r(x,\ y)y'^{n-r} = 0. \qquad (1)$$

If the order of the polynomial F regarded as a function of x and y be ν, then ν is the order of that polynomial $f_r(x, y)$ which has the highest order.

Definition : A curve which satisfies a differential equation of the form (1) is said to be *panalgebraic*. The degree of the curve is said to be n, the degree of the polynomial F in y'. The rank of the curve is said to be ν, the degree of the polynomial F in x and y.

It is evident that the panalgebraic curves are to be regarded as the members of families, each of ∞^1 members : for the integral of equation (1) contains an arbitrary constant.   An algebraic curve may be regarded as a panalgebraic curve of degree zero. The following theorems are fundamental in the theory.

I. If a curve is panalgebraic when regarded as a locus, it is panalgebraic when regarded as an envelope, and conversely.

II. In a family of panalgebraic curves of degree n and rank ν there are n curves which pass through each point of the plane and ν which touch each line.

III. In a family of panalgebraic curves (n, ν) there are $m\nu + n\mu$ which touch a given algebraic curve of order m and class μ.

IV. The points of tangency of the tangents drawn from a given point P to a panalgebraic curve (n, ν) lie on an algebraic curve of order $\nu + n$, of which the given point P is an n-fold point.

V. The tangents drawn at the points of intersection of a panalgebraic curve (n, ν) and a given line p touch an algebraic curve of class $n + \nu$, of which the given line p is an ν-fold tangent.

VI. The cusps of a panalgebraic curve lie on an algebraic curve.

VII. The inflection tangents of a panalgebraic curve touch an algebraic curve.

VIII. The system of all panalgebraic curves is invariant under the group of all contact transformations.

IX. The orthogonal trajectories of a family of panalgebraic curves form a family of panalgebriac curves of the same degree and rank.

X. There exist for panalgebraic curves certain derived algebraic curves analogous to the steinerian and hessian. Hence:

XI. The inflection points of a panalgebraic curve lie on an algebraic curve.

XII. The cuspidal tangents of a panalgebraic curve touch an algebraic curve.

When one considers that some relatively simple panalgebraic curves, such as the epicycloid in case the radii of the two circles are incommensurable, and the herpolhode practically fill completely an entire region of the plane, it is difficult not to be surprised at the simplicity of the results. For example the logarithmic spiral is of degree 1 and rank 1. The application of theorem IV yields the following result : The points of tangency of the tangents drawn from a point P to a logarithmic spiral lie upon a conic which passes through P. The application of the same theorem to Lissajous's curves ($n = 2$, $\nu = 2$) yields : The points of tangency of the tangents drawn from a point P to a Lissajous's curve lie upon a quartic which passes twice through the point P. A similar result obtains for the epicycloid ($n = 2$, $\nu = 4$) except that the locus is a sextic. Two general facts, first the dual property (theorem I), second the invariance under contact transformations (theorem VIII), will surely add to the simplicity and interest of any future detailed theory of panalgebraic curves.

In conclu ion we can not only express our thanks to Professor Loria for his volume on *special* curves and wish it success, but we can, with a pleasure the greater by contrast, recognize in him the founder of a theory of curves more *general* than any heretofore known. We can most heartily join in his closing hope that the present century will bring forth the much desired general theory and classification of transcendental plane curves. It is reasonable to assume that this work of his can not fail to be one of the most important advances toward the full realization of that hope.

EDWIN BIDWELL WILSON.

ÉCOLE NORMALE SUPÉRIEURE, PARIS,
                December 30, 1902.