But generally $\partial\phi/\partial\phi_1$ and $-1/y + px/y^2$ do not vanish, so that (7) reduces to the condition $\partial\phi_1/\partial p_1 = 0$. Hence the theorem :

To the simultaneous equations

$$\phi(x, y, p) = 0, \quad \frac{\partial\phi}{\partial x} + p\frac{\partial\phi}{\partial y} = 0$$

correspond dualistically, the equations

$$\phi_1(x_1, y_1, p_1) = 0, \quad \partial\phi_1/\partial p_1 = 0.$$

The same result would be obtained by operating with the analytic expressions for the most general dualistic transformation. The special transformation (5) was chosen for the sake of simplicity.

UNIVERSITY OF COLORADO,
August, 1903.

HYDRODYNAMIC ACTION AT A DISTANCE.

Vorlesungen über hydrodynamische Fernkräfte nach C. A. BJERKNES'S *Theorie.* Von V. BJERKNES. 2 Bände, 8vo. Leipzig, J. A. Barth, 1900–1902. Bd. I, xvi + 338 pp., 40 figs. Bd. II, xvi + 316 pp., 60 figs.

C. A. BJERKNES is dead. The news is scarcely yet spread over the scientific world. No more fitting time could be found for calling attention to his life-work on hydrodynamic action at a distance. Pupil of Dirichlet at Göttingen, professor of mathematics and physics at the university in Christiania, ardent admirer and follower of Faraday and Maxwell, Bjerknes more than twenty years ago had developed practically to completion a theory which never has received much attention owing to the manner of its publication. That the work is now before the public in a complete, accessible, and easily intelligible form is due to the editorial patience of the son, V. Bjerknes.

The first volume, designed primarily for mathematicians, contains the theoretical development of the mutual actions of pulsating and oscillating spheres immersed in a common incompressible perfect fluid. Fortunately the mathematical analysis is so elementary and so carefully explained that the most meager training amply suffices for its comprehension. In the second volume, which will appeal almost exclusively to physicists, are found the same laws and the formulas governing them, deduced experimentally much in the manner in which the laws of electricity and electromagnetism were discovered and formulated by Coulomb and Faraday. The remarkable fact is that the experimental results are numerically so large. The experiments, which were hastily shown to some of us at the Abel centenary, exhibited the same order of magnitude, the same degree of conclusiveness as the corresponding experiments in the fields of electricity and magnetism. It is fortunate that the firm of Ferdinand Ernecker has undertaken to prepare for the market apparatus similar to that constructed by Bjerknes for use in his private laboratory. A description of the apparatus is found in volume II of the work under review.

To the untrained intuition the phenomena connected with hydrodynamic actions can yield nothing but puzzles. That bodies moving immersed in a common fluid should produce some actions and reactions on one another is readily granted; what sort of actions these may be is not easily divined. For example it is well known that a sphere, free to move and uninfluenced by external forces, may remain at rest in a constant current. That this apparently contradicts all experience and all common sense is due to the imperfection of the spheres and fluids with which we ordinarily come in contact. Under certain circumstances it may even come about that a sphere moves up stream against a strong current. This hydrodynamic paradox, as it is called, has been exhibited far and wide in toys placed upon the market to the mystification of more than children. Even the trained physicist finds it safe to trust rather to analysis than to intuition.

The analysis in the case of arbitrary bodies is far too difficult to follow out. In the case of anchor rings, to which the cosmographic speculations of Lord Kelvin have given such interest, the theory of mutual actions and reactions has not yet reached a satisfactory development, notwithstanding the classic memoir published more than twenty years ago by J. J. Thomson. The present authors when dealing with hydrodynamic actions at a distance restrict themselves for the sake of simplicity to those actions arising from or experienced by spheres pulsating or oscillating with a definite period and separated from one another by a distance great in comparison with the radius of the spheres. If the distance between the spheres be taken as unity, the radius will be a small number, the powers of which may serve as a measure for the different orders of approximation and the different orders of magnitude of the actions which take place. The analysis is carried out in such detail that actions of magnitude down to and including the seventh order are taken into account.

If a sphere

$$(x-a)^2+(y-b)^2+(z-c)^2=d^2$$

pulsate about its fixed center (a, b, c), the radius d and the volume $E=\frac{1}{6}\pi d^3$ are functions of the time t. If the pulsation take place periodically with period τ, the quantity

$$(\dot{E})_m=\pm\sqrt{\frac{1}{\tau}\int_t^{t+\tau}(\dot{E})^2dt}, \quad \text{where} \quad \dot{E}=\frac{dE}{dt},$$

is the quadratic mean of the rate of change of volume and may be regarded as defining the *intensity of pulsation.* The sign $+$ or $-$ is chosen according as the sphere be dilating or contracting at the origin of time.

During the pulsations the particles of the fluid in which the sphere is immersed move radially to and fro. The magnitude and the direction of the velocity of this displacement at any given point of the fluid are derivable from a *velocity potential*

$$\phi=-\frac{\dot{E}}{4\pi r}.$$

It is found convenient to express the rate of cubic dilation $\dot{E}$ as the product of the mean rate $(\dot{E})_m$ and a periodic function $\dot{f}(t)$ which is written as a time derivative for reasons of analogy.

$$\dot{E}=(\dot{E})_m\dot{f}(t).$$

Then

$$\phi=-\dot{f}(t)\frac{(\dot{E})_m}{4\pi r}.$$

Thus the velocity potential is divided into two factors, of which the first, depending on the time, is purely kinematic, while the second is purely geometric.

To a first approximation the velocity potential of a number of pulsating spheres is the sum of the potentials due to the individual spheres

$$\phi=-\sum_i\dot{f}_i(t)\frac{(\dot{E}_i)_m}{4\pi r_i}.$$

The interesting case to consider is that of synchronism in which the periodic functions $\dot{f}_i(t)$ appertaining to the different spheres are identical. The potential again becomes separable into a kinematic and a geometric factor

$$\phi = -\dot{f}(t) \sum_i \frac{(\dot{E}_i)_m}{4\pi r_i}.$$

By comparison with the potential function Φ due to a system of magnetic poles magnetized with intensities I_i,

$$\Phi = + \sum_i \frac{I_i}{4\pi r_i},$$

it appears that the lines of flow set up in a liquid by a system of spheres pulsating with intensities $(\dot{E}_i)_m$ are identical with the lines of force set up by a system of magnetic poles of intensities $I_i = -(\dot{E}_i)_m$ distributed geometrically in the same manner as the spheres.

Further magnetic analogies are immediately derivable by considering a system of two spheres pulsating with equal intensities and opposite phases. The potential is

$$\phi = -\dot{E}\left(\frac{1}{4\pi r_1} - \frac{1}{4\pi r_2}\right).$$

In case the vector $\mathbf{r}_{12}$ joining the center of the first or "positive" sphere to that of the second or "negative" sphere be infinitesimal and the radii of the spheres be infinitesimals of higher order, the system is called a *doublet*. The potential of a doublet takes the form

$$\phi = -\frac{\dot{\mathbf{S}} \cdot \mathbf{r}}{4\pi r^3},^*$$

where $\dot{\mathbf{S}} = \dot{E}\mathbf{r}_{12}$ is by definition the vector *kinematic moment of action* of the doublet, and $\mathbf{r}$ is the radius vector from the center of the doublet to any point in space. The corresponding potential for a "small" magnet is

$$\Phi = +\frac{\mathbf{I} \cdot \mathbf{r}}{4\pi r^3},$$

where **I** is the vector intensity of magnetization.

* The notation for vectors which we adopt here is that of Gibbs. See Vector Analysis by E. B. Wilson, reviewed in the BULLETIN, February, 1902, by A. Ziwet.

Passing to the case of a rigid sphere moving with a velocity $\dot{\mathbf{s}}$ of translation, the potential is found to be

$$\phi = -\frac{1}{2}\frac{d^3}{r^3}\dot{\mathbf{s}} \cdot \mathbf{r}.$$

The vector $\dot{\mathbf{S}} = 2\pi d^3 \dot{\mathbf{s}} = \frac{3}{2}E\dot{\mathbf{s}}$ may be defined as the kinematic moment of action and the potential again takes the form

$$\phi = -\frac{\dot{\mathbf{S}} \cdot \mathbf{r}}{4\pi r^3}.$$

With these conventions appears the remarkable and fundamental fact that the velocity fields set up by a doublet of pulsating spheres and by a sphere moving with translation are identical, provided only that the kinematic moment of action is the same in both cases. In the case of translation the moment of action is three halves the product of the volume by the velocity; for the doublet it is the product of the intensity of pulsation by the distance between the spheres. To treat an oscillating sphere it is necessary to replace the velocity of translation and the moment of action by their mean values.

The parameters a, b, c, d which define the position of a sphere suffer changes of which the velocities are $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{d}$. These parameters and their velocities are divisible into two parts, one slow and progressive, the other fast and oscillating about some mean value. Let

$$a = a^0 + a^1, \quad b = b^0 + b^1, \quad c = c^0 + c^1, \quad d = d^0 + d^1,$$

$$\dot{a} = \dot{a}^0 + \dot{a}^1, \quad \dot{b} = \dot{b}^0 + \dot{b}^1, \quad \dot{c} = \dot{c}^0 + \dot{c}^1, \quad \dot{d} = \dot{d}^0 + \dot{d}^1.$$

As the spheres are not subject to permanent large changes in magnitude, $\dot{d}^0$ is zero and d^0 is constant. a^0, b^0, c^0 change slowly and the velocities $\dot{a}^0$, $\dot{b}^0$, $\dot{c}^0$ are comparable to the velocities of finite masses of matter — the velocities encountered in molar physics. a^1, b^1, c^1, d^1 change rapidly, but remain always small in comparison with d^0, the radius of the sphere. The velocities $\dot{a}^1$, $\dot{b}^1$, $\dot{c}^1$, $\dot{d}^1$ may attain great values and are comparable to the velocities found in molecular physics. Evidently the linear means of the actual velocities $\dot{a}$, $\dot{b}$, $\dot{c}$, d are identical with the linear means of the molar velocities $\dot{a}^0$, $\dot{b}^0$, $\dot{c}^0$, $\dot{d}^0$, and the quadratic means are to a first approximation identical with the quadratic means of the molecular velocities $\dot{a}^1$, $\dot{b}^1$, $\dot{c}^1$, $\dot{d}^1$.

The potential of a system of spheres simultaneously undergoing pulsations and translations is

$$\phi = -\sum_i \frac{\dot{E}_i}{4\pi r_i} - \frac{1}{2}\sum_i \frac{d_i^3}{r_i^3}\dot{\mathbf{s}}_i \cdot \mathbf{r}_i,$$

and may evidently be written in the form

$$\phi = \sum_i (A_i\dot{a}_i + B_i\dot{b}_i + C_i\dot{c}_i + D_i\dot{d}_i),$$

where A, B, C, D are functions of the geometric parameters a, b, c, d and are independent of the velocities $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{d}$. In case of synchronous pulsations and oscillations the geometric functions A, B, C, D may be replaced by their linear mean values, that is, by these values when for a, b, c, d are substituted a^0, b^0, c^0, d^0, and the velocities $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{d}$ may be replaced by their quadratic mean values $\dot{a}_m^1$, $\dot{b}_m^1$, $\dot{c}_m^1$, $\dot{d}_m^1$, which are equal to $\dot{a}_m^1$, $\dot{b}_m^1$, $\dot{c}_m^1$, $\dot{d}_m^1$. The potential then takes on a mean value

$$\phi_m = \sum_i (A_i^0\dot{a}_m^1 + B_i^0\dot{b}_m^1 + C_i^0\dot{c}_m^1 + D_i^0\dot{d}_m^1).$$

The authors next pass to the discussion of an arbitrary velocity field, potential and solenoidal. The function ϕ is developed according to powers of $x - a$, $y - b$, $z - c$ into the form

$$\phi = \phi_0 + \phi_1 + \phi_2 + \cdots + \phi_n + \cdots.$$

Here

$$\phi_1 = \dot{\alpha}(x - a) + \dot{\beta}(y - b) + \dot{\gamma}(z - c),$$

where $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ are the values of the first partial derivatives of ϕ at the point a, b, c.

$$\phi_2 = \tfrac{1}{2}\dot{\alpha}_\alpha(x - a)^2 + \tfrac{1}{2}\dot{\beta}_\beta(y - b)^2 + \tfrac{1}{2}\dot{\gamma}_\gamma(z - c)^2 + \dot{\beta}_\gamma(y - b)(z - c) + \text{etc.},$$

where $\dot{\alpha}_\alpha$, $\dot{\beta}_\beta$, $\dot{\gamma}_\gamma$, $\dot{\beta}_\gamma$, etc., are values of the second partial derivatives of ϕ at the point a, b, c. In vector form the expansion may be written

$$\phi = \phi_0 + \mathbf{r} \cdot \nabla\phi + \mathbf{r} \cdot \nabla\nabla\phi \cdot \mathbf{r} + \cdots,$$

where

$$\dot{\nabla} \cdot \nabla\phi = 0.$$

The term $\phi_1 = \mathbf{r} \cdot \nabla\phi$ defines a parallel field. The equipotential surfaces are planes; and the lines of flow, straight lines. The second term defines what Bjerknes calls a deformation field. The equipotential surfaces are quadrics to which the lines of flow are everywhere perpendicular.

After these preliminary considerations the authors take up the general kinematic actions and reactions between a sphere and a current. The potential due to the motion of a sphere immersed in a fluid originally at rest is divided into two parts; the internal, that of the motion of the sphere itself, and the external, that due to the motion induced in the fluid by the motion of the sphere. In the case of simple pulsations or oscillations these potentials are as given above. But as the field of the stream acts on the sphere, the relations become far more complicated and require a more elaborate treatment. If, in the general case, the interior potential of the sphere be

$$\Phi = -\tfrac{1}{2}\dot{d} + \Phi_1 + \cdots + \Phi_n + \cdots,$$

the exterior is

$$\phi = -\frac{d^2}{r}\dot{d} - \tfrac{1}{2}\frac{d^3}{r^3}\Phi_1 - \cdots - \frac{n}{n+1}\frac{d^{2n+1}}{r^{2n+1}}\Phi_n - \cdots.$$

The potential of a stream in which a sphere at rest is immersed is likewise divided into two parts. If the original, that is to say incident, stream be defined by the potential

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_n + \cdots,$$

the total resulting stream has the potential

$$\phi = \phi_0 + \left(1 + \tfrac{1}{2}\frac{d^3}{r^3}\right)\phi_1 + \cdots + \left(1 + \frac{n}{n+1}\frac{d^{2n+1}}{r^{2n+1}}\right)\phi_n + \cdots,$$

of which

$$\tfrac{1}{2}\frac{d^3}{r^3}\phi_1 + \cdots + \frac{n}{n+1}\frac{d^{2n+1}}{r^{2n+1}}\phi_n + \cdots$$

is the *reaction* or *reflex* potential arising from the obstructing presence of the sphere at rest in the stream. It results that for a sphere moving in a current the total external potential takes the form

$$\phi = -\frac{d^3}{r}\dot{d} - \sum_{n=1}^{\infty}\frac{n}{n+1}(\Phi_n - \phi_n) + \sum_{n=1}^{\infty}\phi_n,$$

of which the first term arises directly from the pulsation, the second gives the difference between the potentials of the sphere and of the current, and the third is due to the incident stream. This formula serves as the starting point of a number of applications to the motion of spheres advancing and pulsating in parallel or deformation fields. The cuts which accompany the text are admirably drawn and instructive.

To pass to dynamic considerations, use is made of the formula for the pressure p at any point of the fluid, expressed in terms of the velocity potential ϕ, the pressure P which would exist if the system were at rest, and the density q,

$$p = P - q\frac{\partial\phi}{\partial t} - \tfrac{1}{2}q\nabla\cdot\nabla\phi.$$

By somewhat lengthy analysis, filled with approximations, the authors succeed in establishing the formula for the total (approximate) force acting upon the sphere,

$$\mathbf{F} = -q\frac{d}{dt}\{E(\tfrac{1}{2}\dot{\mathbf{s}} - \tfrac{3}{2}\dot{\sigma})\} - q\dot{E}\dot{\sigma} - \tfrac{3}{2}qE\{(\dot{\mathbf{s}} - \dot{\sigma})\cdot\nabla\nabla\phi\}.$$

It is to be noted that $\dot{\sigma}$ is the velocity of the stream ϕ, and that of the sphere. This value for $\mathbf{F}$ is *exact* if the field ϕ be a parallel field, a deformation field, or a combination of both. In other cases the error depends on the degree of divergence of the field from the nearest possible combination of a parallel and a deformation field. In practice the divergence is so slight that the expression above differs from the exact value of $\mathbf{F}$ by a quantity of the eighth order at most and may consequently be regarded as quite satisfactory.

The equation of motion for a sphere of mass M immersed in the fluid and acted upon by an external force $\mathbf{f}$ is

$$M\mathbf{a} = M\ddot{\mathbf{s}} = \mathbf{F} + \mathbf{f}.$$

The integration of this equation gives the complete theory of hydrodynamic action at a distance. The special cases which arise, according as some of the terms of $\mathbf{F}$ are missing or as ϕ has a special form, yield interesting results easily confirmable by experience. These the authors consider first.

As the whole trend of the work is to exhibit the analogies to electricity and magnetism with especial regard to the point of view of Maxwell, there is great attention paid to the medium, that is, the liquid in which the actions take place. The reciprocal of the density, or the specific volume as it is also called, carries the name of *mobility* and corresponds to the polarizability of the mediums in electricity and magnetism. The velocity is multiplied by the density q to give the *intensity of the field* in the fluid. The kinematic moment of action of a sphere is likewise multiplied by the density q to obtain the *dynamic moment of action*. The velocity potentials, ϕ in the fluid and Φ in the sphere, are replaced by the potentials of the field intensity, $q\phi$ and $Q\Phi$ respectively (where Q is the density of the sphere).

The total force $\mathbf{F}$ is separated into two parts. The force of *hydrodynamic induction*

$$\mathbf{F}_{\text{ind}} = -q\frac{d}{dt}\{E(\tfrac{1}{2}\dot{\mathbf{s}} - \tfrac{3}{2}\dot{\sigma})\}$$

is a perfect time derivative and derives its name from electric and magnetic analogies. The force of *hydrodynamic energy*

$$\mathbf{F}_{\text{en}} = -q\dot{E}\dot{\sigma} - \tfrac{3}{2}qE\{(\dot{\mathbf{s}} - \dot{\sigma}) \cdot \nabla\nabla\phi\}$$

differs in no essential way from the external force $\mathbf{f}$. The first integration of the equation of motion gives

$$\dot{\mathbf{s}} = -\frac{qE}{M}(\tfrac{1}{2}\dot{\mathbf{s}} - \tfrac{3}{2}\dot{\sigma}) + \frac{1}{M}\int_0^t (\mathbf{F}_{\text{en}} + \mathbf{f})\,dt.$$

The last term is the velocity generated by the energy forces external and internal, and will be denoted by $\dot{\mathbf{s}}_{\text{en}}$. Then

$$\dot{\mathbf{s}} = \frac{Q}{Q + \frac{1}{2}q}\dot{\mathbf{s}}_{\text{en}} + \frac{\frac{3}{2}q}{Q + \frac{1}{2}q}\dot{\sigma}.$$

As the greater number of the fields with which the authors deal are set up by pulsating and oscillating spheres, the value of $\dot{\sigma}$ oscillates about zero. This gives rise to a new nomenclature. The first term in $\dot{\mathbf{s}}$, above written, is called the *permanent* velocity; the second, the *temporary*. Subsequently the force of energy is likewise subdivided into the permanent force of energy $\mathbf{F}_{p\text{-en}}$, and the temporary

$$\mathbf{F}_{t\text{-en}} = -\tfrac{3}{2}q\frac{q - Q}{Q + \frac{1}{2}q}E\dot{\sigma} \cdot \nabla\nabla\phi.$$

It is evident that no temporary force of energy exists, unless the densities of the liquid and sphere differ. The force of induction is divided into the self-induction

$$\mathbf{F}_{s\text{-ind}} = -q\frac{d}{dt}\{E(\tfrac{1}{2}\dot{\mathbf{s}})\},$$

which depends not at all on the stream but only on the sphere, and the foreign induction

$$\mathbf{F}_{f\text{-ind}} = +q\frac{d}{dt}\{E(\tfrac{3}{2}\dot{\sigma})\},$$

dependent upon the velocity of the stream and the volume of the sphere.

After a discussion, for about eighty pages, of the results obtainable from this subdivision of the total force into four partial forces, there follows a short chapter on rotary effects. As the fluid is frictionless, the individual spheres can not be set into rotation about their axes; but if a doublet or an oscillating sphere be considered, the line of centers of the spheres composing the doublet or the line of oscillation of the sphere is subject to change. The moment of rotation due to the force of energy is very simple,

$$-\tfrac{3}{2}qE\dot{\mathbf{s}}\times\dot{\sigma},$$

and that due to the induction is

$$-q\frac{d}{dt}\{E(\mathbf{r}-\mathbf{r}_0)\times(\tfrac{1}{2}\dot{\mathbf{s}}-\tfrac{3}{2}\dot{\sigma})\},$$

where $\mathbf{r}-\mathbf{r}_0$ is the vector drawn between the positions of the center of the sphere at two successive instants.

To this point the question has been to treat the mutual influence of a stream ϕ and a sphere. If now the stream be the result of the motions of other spheres the expressions above found for forces and moments will give the actions of the other (distant) spheres upon the sphere in question. As the stream ϕ must be a composition of a parallel and a deformation field if the formulas obtained may be considered as exact, and as the streams set up by other spheres do not in general come under this restricted category, the formulas for forces and moments are only approximate. The error, however, is very small — of the eighth order at most. The formulas are accurate down to and including the seventh order. Thus if the ratio of the radius of the spheres to the distance between them be of the magnitude one tenth, the errors are comparable to one one-hundred-millionth.

Properly speaking, the self-induction is not an action at a distance. But the foreign induction is; and is of the second and third orders. The permanent forces of energy are of the second, third, and fourth orders. The temporary forces of energy are of the fifth, sixth, and seventh orders and are called actions of higher order to distinguish them from the others which are all stronger.

Everything is now ready for discussing the mechanics of a system of spheres advancing and pulsating in a fluid — the mechanics which would appear to a person situated in the fluid, and ignorant of its presence or unwilling to take cognizance of the medium as a transmitter of actions which follow the laws of our ordinary mechanics. It is found that for the forces of lower order, that is, for the foreign induction and the permanent forces of energy :

1. The law of inertia holds intact.

2. The second law, that the impulse is in the line of the force and proportional in magnitude to it, also holds.

3. The forces due to distant spheres may be compounded according to the parallelogram law.

4. The forces are independent of the velocity or acceleration of the point of application.

5. The principle of equal action and reaction holds for the forces of energy, and although the action and reaction arising from the induction are not equal and opposite, yet the inequalities influence mainly the motion of the individual spheres and produce no sensible effects on the "molar" motions of systems of spheres in large number, provided only that the distance between the constituent spheres be large in comparison to their radii.

6. The hydrodynamic actions are conservative in nature.

From this it results that to an observer situated in the fluid and made up of spheres as we are made up of atoms the system of mechanics of molar masses of spheres would appear identically to follow the laws of Newton. In reality there would be imperceptible differences, due first to the statement under caption 5 and secondly to the actions of forces of higher orders, which fail to obey most of the six captions stated above.

Of the experimental effects observable upon spheres we have mentioned none directly. The formulas quoted above enable us to infer many facts of more or less obvious analogy, with electric and magnetic phenomena:

1. The self-induction causes a sphere to move more slowly than in free space by the ratio $Q : (Q + \frac{1}{2}q)$.

2. If the external force $\mathbf{f}$ be chosen so as at each instant to equilibrate the internal force of energy $\mathbf{F}_{en}$, the total motion is due to induction, the induced field is a parallel field, and the sphere moves with the velocity $\frac{3}{2}q : (Q + \frac{1}{2}q)$ of the field.

If by the axis of a doublet or oscillating sphere be meant the direction of the vector kinematic movement of action $\mathbf{S}$, then

3. Spheres pulsating in the same phase attract, in opposite phases repel each other.

4. A pulsating sphere and an oscillating sphere whose axis is directed toward the center of the pulsating attract; if the axis is directed in the opposite direction they repel each other.

5. If the axis of the oscillating sphere does not pass through the center of the pulsating sphere, this latter moves in the direction of the axis of the former which at the same time turns its axis toward the pulsating sphere.

6. If two oscillating spheres have parallel axes, attraction takes place in case the directions are the same; repulsion in case they are opposite.

7. If the axes of two oscillating spheres lie in the same line and have the same direction the phenomenon is repulsion; if opposite directions, attraction.

And in general when the axes are arranged arbitrarily the spheres undergo translations and rotations according to the laws which would govern magnets if the signs of the forces were all changed.

To consider the temporary forces of energy, which are of higher order, it is necessary to mask all the actions of lower order. By a glance at the formulas it appears that the actions of lower order vanish if the sphere upon which they are supposed to act is at rest or simply pulsating, and that the temporary actions are increased by increasing the difference in densities between the sphere and the liquid. An investigation shows that:

1. An oscillating sphere attracts a sphere at rest if the sphere be heavier than the liquid, but repulses it if it be lighter. The force is of the seventh order.

2. For a pulsating system the result is the same except that the order of the force is the fifth.

3. Two spheres heavier than the liquid and pulsating with opposite phases repulse each other at long distances and attract at short distances. There is an intermediate position of equilibrium. Spheres lighter than the liquid and pulsating in the same phase attract each other at long distances and repulse at short distances.

The analogies which suggest themselves during the course of the work on hydrodynamic actions at a distance are many. The fact that the motions must be divided into visible and invisible recalls the system of mechanics of Hertz, in which hidden masses and hidden motions are freely introduced to make the system come under the canonical heading. The behavior of the temporary forces calls to mind the laws of molecular actions assumed by theorists for constructing the kinetic theory of gases. All assume a force function which gives attraction at great distances and repulsion at small. The order of the force differs with different authors. The fact that forces of all orders come into play gives a chance to draw analogies with the behavior of matter in regard to gravitation, adhesion and cohesion. But of all the analogies the most fruitful seems to be that which relates to electricity or magnetism. The correspondence seems to be nearly complete. The pulsating sphere corresponds to an electric charge or a magnetic pole; the oscillating sphere, to a doublet or short magnet. The correspondence is, however, negative; where attraction occurs in electricity and magnetism, repulsion is found in hydrodynamics. Why this should be is not known. Perhaps it is only chance, or perhaps it is due to some subtle properties of matter and electricity which hitherto have evaded discovery.

The following table gives in résumé the correspondences which exist between the quantities occurring in hydrodynamics and in electricity or magnetism:

	Electric Medium.	Fluid.
q:	Ponderomotive constant of activity.	Density.
k:	Polarizibility.	Mobility.
$\dot{E}$:	Free pole intensity.	Kinematic intensity of pulsation.
$q\dot{E}$:	True pole intensity.	Dynamic intensity of pulsation.

$\mathbf{S}$:	True electric or magnetic moment.	Kinematic moment of action.
$q\mathbf{S}$:	Free electric or magnetic moment.	Dynamic moment of action.
$q\dot{\mathbf{S}}$:	Electric or magnetic field intensity.	Hydrodynamic field intensity.
$\dot{\mathbf{S}}$:	Electric or magnetic polarization.	Velocity.

Having completed our account of the voluminous work under review, it is well to ask what use, what gain accrues to us. In the first place the sole fact that here is a large field of hydrodynamics worked out in detail, analytically and experimentally, is sufficient cause for congratulation. The problems of hydrodynamics which have been carried through to the end are few. The subject and the presentation are often difficult. The theory of the tides, the theory of resistance offered to boats are examples. The theory of C. A. Bjerknes has its difficult points; but so clearly is all explained that the reader does not perceive the burden of the difficulties. At the present state of our knowledge the electric and magnetic analogies seem to be more interesting than valuable. This the authors recognize with their characteristic frankness and no small profit may be obtained by considering the justness with which they estimate and discuss their work.

Another great value and interest in the work lies in its relation to cosmographical speculations. Some years ago one of the presidents of the British Association said in his annual address that since the scientific world, especially the English, had taken such a fancy to devising schemes on which to explain the universe, there would be large interest even if small utility in collecting and comparing the different suggestions of which practically all are of hydrodynamic origin. First is Kelvin's vortex atom and it is doubtless responsible for the rest. Someone pointed out that one vortex anchored at each end to the boundary surface of the universe would answer all purposes. Hicks brought out a "bubble" theory according to which matter was to be hollow spaces in the ether. The development of the theory may be read with great difficulty in McAulay's Utility of Quaternions in Physics. Later Pearson produced an "ether-squirt" theory. An atom, according to him, consists of a point discharging ether into space with a pulsating rate. The mathematical investigations may be found in the *Cambridge Transactions* and in the *American Journal of Mathematics*. They are not easy to read. C. A. and V. Bjerknes do not pretend to have laid the foundations of the universe. They have treated a broad problem in hydrodynamics and treated it clearly, completely, systematically. To those who would read with any ease the developments of Hicks's and Pearson's theories the work of Bjerknes is a practical necessity; to mathematician and physicist alike it is interesting; and had it appeared twenty years ago when first complete, it would doubtless have attracted much more attention and possessed a much greater influence than now.

EDWIN BIDWELL WILSON.

PARIS,
June, 1903.

SHORTER NOTICES.

Vorlesungen über Geschichte der Trigonometrie. Von DR. A. VON BRAUNMÜHL. Zweiter Teil. Leipzig, B. G. Teubner, 1903. xi + 264 pp.

THE first part of von Braunmühl's History of Trigonometry was reviewed in the BULLETIN, volume 6 (1900), page 404. The second part is fully up to the standard of the first. The completed work will take its place as the fullest and most authoritative history of trigonometry that we have. Cantor's great work, Vorlesungen über Geschichte der Mathematik, comes down only to the year 1758, so that the present history of trigonometry covers nearly one hundred and fifty years of this science which have never been treated before with any degree of thoroughness.

The author begins the second part with the history of logarithms. In this connection he gives an account of John Speidell, the author of the first table of natural logarithms. Hitherto, German writers have overlooked Speidell. De Morgan's interesting account of him, given in the article "Tables" in the English Cyclopœdia, does not seem to have been used by von Braunmühl. He examined the reprint of Speidell's work, the New Logarithmes, that is given in Maseres's Scriptores Logarithmici, volume 6, page 713. Maseres reprinted from the edition of 1628, yet von Braunmühl, unaware of this fact, refers in a footnote, page 26, to a remote source