ON LAMÉ'S SIX EQUATIONS CONNECTED WITH TRIPLY ORTHOGONAL SYSTEMS OF SURFACES.

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LAMÉ* has shown for a triply orthogonal system of surfaces given by the parameters ρ, ρ_1, ρ_2 that if the square of the element of length is given by $ds^2 = H^2 d\rho + H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2$, where H, H_1, H_2 are certain functions of ρ, ρ_1, ρ_2, then H, H_1, H_2 must satisfy the following system of equations:

$$\frac{\partial^2 H}{\partial\rho_1 \partial\rho_2} = \frac{1}{H_1}\frac{\partial H}{\partial\rho_1}\frac{\partial H_1}{\partial\rho_2} + \frac{1}{H_2}\frac{\partial H}{\partial\rho_2}\frac{\partial H_2}{\partial\rho_1} \qquad (1)$$

and two others of the same type; (2), (3)

$$\frac{\partial}{\partial\rho_1}\left(\frac{1}{H_1}\frac{\partial H}{\partial\rho_1}\right) + \frac{\partial}{\partial\rho}\left(\frac{1}{H}\frac{\partial H_1}{\partial\rho}\right) + \frac{1}{H_2^2}\frac{\partial H}{\partial\rho_2}\frac{\partial H_1}{\partial\rho_2} = 0 \qquad (4)$$

with two others of this type. (5), (6)

Also if V is a function of x, y, z for which

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

he has shown that

$$\frac{\partial}{\partial\rho}\left(\frac{H_1 H_2}{H}\frac{\partial V}{\partial\rho}\right) + \frac{\partial}{\partial\rho_1}\left(\frac{H_2 H}{H_1}\frac{\partial V}{\partial\rho_1}\right) + \frac{\partial}{\partial\rho_2}\left(\frac{H H_1}{H_2}\frac{\partial V}{\partial\rho_2}\right) = 0.$$

If the system of coordinates ρ, ρ_1, ρ_2 is isothermal, this equation must be satisfied by $V = \rho$, or by $V = \rho_1$, or by $V = \rho_2$. Hence $H_1 H_2 / H = Q^2$, where Q is a function of ρ_1 and ρ_2 only. Similarly $H_2 H / H_1 = Q_1^2$, and $H H_1 / H_2 = Q_2^2$, where Q_i is a function not involving the variable ρ_i. Hence $H = Q_1 Q_2$, $H_1 = Q_2 Q$, $H_2 = Q Q_1$. The six equations given above transform into six in the variables Q. Lamé † gives a solution of

* Leçons sur les coordonées curvilignes (1859), pp. 76, 78.
† Loc. cit., p. 99.

these equations by first finding a solution of equations (1), (2), (3) and using this to solve the remainder. He makes the statement that his solution of the first three equations is the most general possible * ; this statement is obviously inaccurate and it seems of interest to give a complete solution of the equations. The solution of the first three equations, or rather a comparison of two different solutions, leads to a curious result in the theory of elimination.

Equation (1) becomes

$$Q\frac{\partial Q_2}{\partial \rho_1}\frac{\partial Q_1}{\partial \rho_2} = Q_1\frac{\partial Q_2}{\partial \rho_1}\frac{\partial Q}{\partial \rho_2} + Q_2\frac{\partial Q_1}{\partial \rho_2}\frac{\partial Q}{\partial \rho_1}, \tag{7}$$

with similar expressions for (2) and (3). Multiply (7) by $\partial Q_1/\partial \rho$, and the transformed expression for (3) by $\partial Q_1/\partial \rho_2$, and add. The result is

$$Q_1\left[\frac{\partial Q_2}{\partial \rho}\frac{\partial Q}{\partial \rho_1}\frac{\partial Q_1}{\partial \rho_2} + \frac{\partial Q_1}{\partial \rho}\frac{\partial Q_2}{\partial \rho_1}\frac{\partial Q}{\partial \rho_2}\right] = 0.$$

Hence unless all the Q's vanish identically

$$\frac{\partial Q_2}{\partial \rho}\frac{\partial Q}{\partial \rho_1}\frac{\partial Q_1}{\partial \rho_2} + \frac{\partial Q_1}{\partial \rho}\frac{\partial Q_2}{\partial \rho_1}\frac{\partial Q}{\partial \rho_2} = 0. \tag{8}$$

Assume that none of the derivatives in (8) vanish identically and write

$$K = -\frac{\partial Q}{\partial \rho_1}\Big/\frac{\partial Q}{\partial \rho_2}, \quad K_1 = -\frac{\partial Q_1}{\partial \rho_2}\Big/\frac{\partial Q_1}{\partial \rho}, \quad K_2 = -\frac{\partial Q_2}{\partial \rho}\Big/\frac{\partial Q_2}{\partial \rho_1}.$$

Equation (8) becomes $KK_1K_2 = 1$, where K_i is a function not involving ρ_i. By taking logarithms and differentiating with respect to two of the variables ρ, it is easy to prove that the most general values for the K's are

$$K = \frac{a_2}{a_1}, \quad K_1 = \frac{a}{a_2}, \quad K_2 = \frac{a_1}{a},$$

where a is a function of ρ only and similarly a_1 and a_2 are functions of ρ_1 and ρ_2 alone respectively.

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* Loc. cit., p. 100, line 23.

If we put $\int a d\rho = \sigma$, with similar values for σ_1 and σ_2, we deduce that Q is a function of $\sigma - \sigma_2$ only, Q_1 a function of $\sigma_2 - \sigma$ only, and Q_2 a function of $\sigma - \sigma_1$ only. Substitute these values in (7) and it readily reduces to

$$\frac{Q}{Q'} + \frac{Q_1}{Q_1'} + \frac{Q_2}{Q_2'} = 0.$$

Hence

$$\frac{Q}{Q'} = n(\sigma_1 - \sigma_2), \quad \frac{Q_1}{Q_1'} = n(\sigma_2 - \sigma), \quad \frac{Q_2}{Q_2'} = n(\sigma - \sigma_1),$$

where n is a constant, and therefore

$$Q = c(\sigma_1 - \sigma_2)^{1/n}, \quad Q_1 = c_1(\sigma_2 - \sigma)^{1/n}, \quad Q_2 = c_2(\sigma - \sigma_1)^{1/n}, \quad (A)$$

where c, c_1, c_2 are constants. This is the most general solution, provided none of the derivatives in (8) vanish. If, however, for example, $\partial Q_2/\partial\rho_1 = 0$, then either $\partial Q_2/\partial\rho = 0$, or $\partial Q/\partial\rho_1 = 0$, or $\partial Q_1/\partial\rho_2 = 0$. Equation (7), however, shows that $\partial Q_2/\partial\rho_1 = 0$ implies either $\partial Q/\partial\rho_1 = 0$ or $\partial Q_1/\partial\rho_2 = 0$. $\partial Q_2/\partial\rho_1 = 0$, $\partial Q_1/\partial\rho_2 = 0$ lead to the solution

$$Q_1 = f(\rho), \quad Q_2 = cf(\rho), \quad Q = \phi(\rho_1\ \rho_2), \qquad (B)$$

where f and ϕ are arbitrary functions of their arguments and c is a constant. $\partial Q_2/\partial\rho_1 = 0$, $\partial Q/\partial\rho_1 = 0$ lead to the solution

$$Q_2 = f(\rho), \quad Q = \phi(\rho_2), \quad Q_1 = F(Q,\ Q_2), \qquad (C)$$

where F is homogeneous and of unit degree in Q_1, Q_2. These types (A), (B), (C) are the only three types of solution.

We now proceed to the integration of the equations in a different manner. Write $\log Q = \lambda$, $\log Q_1 = \lambda_1$, $\log Q_2 = \lambda_2$. Equations (1), (2), (3) become

$$\frac{\partial\lambda_2}{\partial\rho_1}\frac{\partial\lambda_1}{\partial\rho_2} = \frac{\partial\lambda_2}{\partial\rho_1}\frac{\partial\lambda}{\partial\rho_2} + \frac{\partial\lambda}{\partial\rho_1}\frac{\partial\lambda_1}{\partial\rho_2} \qquad (9)$$

and two similar equations.

Write $\lambda_1 - \lambda_2 = \omega$, $\lambda_2 - \lambda = \omega_1$, $\lambda - \lambda_1 = \omega_2$ and (9) becomes

$$\frac{\partial\omega_1}{\partial\rho_1}\frac{\partial\omega_2}{\partial\rho_2}-\frac{\partial\omega_1}{\partial\rho_2}\frac{\partial\omega_2}{\partial\rho_1}=0,$$

or

$$J\begin{pmatrix}\omega_1 & \omega_2\\ \rho_1 & \rho_2\end{pmatrix}=0.$$

Hence the equation implies the existence of a relation, $f(\omega_1, \omega_2, \rho)=0$. Exactly similarly there are relations

$$f_1(\omega_2, \omega, \rho_1)=0, \quad f_2(\omega, \quad \omega_1, \rho_2)=0.$$

From these equations, together with $\omega+\omega_1+\omega_2=0$, it is easy to deduce that either ρ, ρ_1, ρ_2 do not any of them occur explicitly in f, f_1, f_2 or if, for example, ρ occurs in f, it is easy to prove that either ρ_1 is absent from f_1, or ρ_2 from f_2. The latter case is thus reduced to the former, for if one relation of the type $f(\omega, \omega_1)=0$ exist, then two others of that type also exist in virtue of the relation $\omega+\omega_1+\omega_2=0$.

Substituting for the ω's in terms of the Q's, we immediately deduce that the solution is equivalent to the statement that a homogeneous relation $F(Q, Q_1, Q_2)=0$ exists among the Q's.

Combining the two solutions we have the following theorem connected with the theory of elimination:

Let $F(Q, Q_1, Q_2)=0$ *be any homogeneous relation. It is possible to express* Q *as a function of two variables* ρ_1 *and* ρ_2, Q_1 *as a function of two variables* ρ_2 *and* ρ, *and* Q_2 *as a function of the variables* ρ *and* ρ_1, *in two cases only :*

(*A*) *If* F *is of the form* $aQ^n+a_1Q_1^n+a_2Q_2^n$, *where* a, a_1, a_2 *are constants.**

(*C*) *If* F *is general, and e. g.* Q *is a function of* ρ_1 *only, and* Q_1 *a function of* ρ *only.*

It is not difficult to complete the solution of the equations (1), $\cdots$, (6). It may readily be shown that for case (*A*) (4), (5), and (6) are not satisfied unless $n=\frac{1}{2}$, and then

$$\sigma=A\wp\left(\frac{\rho+a}{\sqrt{c}}, g_2, g_3\right)+B,$$

$$\sigma_1=A\wp\left(\frac{\rho_1+a_1}{\sqrt{c_1}}, g_2, g_3\right)+B,$$

*This includes (*B*).

$$\sigma_2 = A\wp\left(\frac{\rho_2 + a_2}{\sqrt{c_2}}, g_2, g_3\right) + B,$$

where the a's, the g's, A and B are constants.

For case (c) the complete solution is

$$Q_1 = a \operatorname{cosec}(b\rho_2 + c), \quad Q_2 = a' \operatorname{cosec}(b'\rho_1 + c'),$$
$$Q = A[\operatorname{cosec}^2(b\rho_2 + c) - \operatorname{cosec}^2(b'\rho_1 + c')]^{\frac{1}{2}},$$

where A, a, a', b, b', c, c' are constants such that

$$a^2b'^2 + a'^2b^2 = 0.$$

In case (B)

$$Q_1 = \frac{1}{a\rho + b}, \quad Q_2 = \frac{k}{a\rho + b},$$

where a, b, k are constants, and Q is a function of ρ_1 and ρ_2 which satisfies the equation

$$\left(k^2\frac{\partial^2}{\partial\rho_2^2} + \frac{\partial^2}{\partial\rho_1^2}\right)\log Q + a^2Q^2 = 0.$$

Of these three types of solution, the first is the same as that given by Lamé.* He gives it in different form, and his method of obtaining it is different. He falls into the error of imagining that the most general solution of the *first* three of his equations corresponds to the case of $n = \frac{1}{2}$, and it happens that this error is largely corrected because the *second* three equations require this limitation in case (A); case (B), however, escapes his notice.

The surfaces corresponding to the three solutions are readily obtained. (A) gives a system of confocal quadrics, and (C) a system of confocal spheroids with their axial planes. (B) gives a system of concentric spheres, and the conical surfaces obtained by joining the common centre to any set of isothermal lines on one of them.

BRYN MAWR,
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* Loc. cit., p. 104.