A few words should be added about the contents of the book. The usual range of topics is included, without neglecting applications in mechanics and with the addition of brief but suitable chapters upon ordinary differential equations and upon tangent lines and planes in space. Special attention is given to the parametric representation of curves. An unfortunate omission will be noted under the topic of differentials. It is not proved nor even remarked that the differential and increment of a function $f(x)$ differ from each other by an infinitesimal of higher order, although the proof of this important fact would occupy only a few lines. The application of differentials to the approximate computation of small increments of $f(x)$ is simultaneously excluded. The corresponding omissions in the case of the differential of a function of two or more variables are especially to be regretted, for the differential (§§ 136, 137) is left devoid of significance when the variables are independent.

In conclusion, generous recognition should be accorded to the care which has been bestowed upon the work. At many points improvements over our current text-books will be noticed, not in themselves sufficiently important to dilate upon but having together great cumulative force. As an instance, I shall cite the inclusion of a real proof that two functions which have a common derivative can differ only by a constant. The introductory chapter on the concepts continuity, function, and limits can also be especially commended, and the chapters on series and the expansion of functions. I know of no work which has greater promise of success in our college classes.

EDWARD B. VAN VLECK.

GENEVA,
September 20, 1905.

THE FOUNDATIONS OF SCIENCE.

Wissenschaft und Hypothese. Von HENRI POINCARÉ. Autorisierte deutsche Ausgabe mit erläuterenden Anmerkungen von F. und L. LINDEMANN. Leipzig, B. G. Teubner, 1904. xvi + 342 pp.

NOT logical enough for the logician, not mathematical enough for the mathematician, not physical enough for the physicist, not psychological enough for the psychologist, nor metaphysical enough for the metaphysician, Poincaré's Science and Hypothesis can hardly give the satisfaction of finality to any one; and yet it probably comes nearer to satisfying the requirements of all these classes of investigators than any single book of our acquaintance. To the author this antithesis ought to be the source of the greatest gratification, in view of his position that a theory is never of greater service to science than when it breaks down. For this little book breaks down constantly, although it is full of brilliant suggestion on every topic. Were it possible to find from the pens of the leading logicians, psychologists, physicists, and metaphysicians such works as this from the pen of a supreme mathematician, there would be at hand the material for reaching a satisfactory and unified theory of science — provided, of course, that such a theory is possible. But is it possible? And even if, to follow Bôcher's happy phrasing,* some method of successive approximation is really leading us toward absolute rigor and unity, why should we assume that we shall ever reach the limit, or even that the limit exists?† May we not be forever forced to rest content with a mere feeling of convergence —slower or faster, and very likely asymptotic at best? If so, works like the present will always remain necessary and highly useful, perhaps more useful than those which give a more consistent theory which is less true to natural limitations.

In Part I the author analyzes the concepts of number and magnitude. He affirms (page 17) that the principle of mathematical induction and it alone can teach us something new. The argument is plausible but seems at times too narrow, founded too much on the principle of arithmetization in its most meager form, and too closely associated with the syllogism as the fundamental method of reasoning. In geometry, if not in algebra, one feels conscious of other methods of proof, other methods of passing from the particular to the general and conversely. Furthermore the symbolic logicians have pointed

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* BULLETIN, vol. 11 (1904), p. 120.

† It should be remembered that in order to classify and otherwise deal with experience, science is bound to idealize. For different purposes different idealizations are convenient, and these may sometimes be mutually contradictory without introducing great inconvenience. Take, for instance, the conception of infinite. The idea of finiteness is definite. When, however, we negative the idea, we are often led to harmless contradictory statements. Thus "the portion of the plane which is at infinity is a line" and "the portion of the plane which is at infinity is a point" cannot both be true, but are in constant use as different idealizations, and introduce no difficulty when each is kept in its own field. Recently a great deal of disagreement has arisen in trying to relate the transfinite ordinals and cardinals — perhaps they are harmlessly contradictory.

out the fact that other modes of reasoning than the syllogistic are valid and essential parts of our logical machine. In the subsequent discussion of irrational numbers and the continuum, the author's point of view comes emphatically to the foreground, namely, that it is physics which is the real thing, that our mathematics is purely arbitrary and a matter of convenience alone, always guided however by our sense impressions and hence somewhat limited in its arbitrariness as far as practical results are concerned. He says: "The mind does not use its creative faculty except when experience forces it to do so" (page 30.) Although a pure mathematician, imbued to a less extent than is Poincaré with the idea that it is applied mathematics which is essentially important and that pure mathematics is only relatively so, might dissent from this opinion, there is contained in it at least a highly valuable pedagogic suggestion. The main difficulty in getting students interested in pure mathematics consists in overcoming their inability to see what it is all for.

Part II on space and its relations to geometry and experience is one of the most original and suggestive in the book. A few quotations will serve to exhibit the author's point of view, if not his argument: "The geometric axioms are neither a priori synthetic judgments nor experimental facts. They are merely conventions, or in other words only disguised definitions. Geometry is not an experimental science, though experience guides us in setting up the axioms. The question whether the euclidean geometry is the *true* geometry has no more sense than whether the metric system of measurement is the true system (pages 51, 73, 138)." Bôcher has recently discussed this point of view, and come to conclusions with which we are in substantial agreement.* Poincaré goes on to analyze with care the relations between the geometric space which we imagine on account of its convenience and the various spaces which our senses of sight, touch, and motion afford. The characteristics of homogeneity and isotropy, for instance, are not due to any individual one of these spaces, but arise from studying the laws under which our sense impressions succeed one another. Here the author's point of view is largely that of the psychologist. It may be well to note that the French mathematicians are not content with the purely logical development of geometry from a set of more or less artificial postulates. They insist that psychology

* Loc. cit., p. 124.

be taken into account. Picard writes: * "In these questions, the geometric logicians appear to pay too little attention to psychology, and to the information furnished us by uncivilized peoples." (He is referring primarily to the concept of number, to which the argument applies with certainly no greater force than to geometry.)

This question of how far psychology of individuals or of races should be heeded is very interesting and brings up the whole matter of what the axioms or postulates are for and in how far they actually characterize real space. The mathematician is wont to say that, if the system of postulates is complete, it characterizes completely the (mathematical or logical) space in question. By this he means that he has defined, whether by postulates or nominally, a set of elements connected by certain relations and capable of being put into one to one correspondence with the elements of space in such a way as to leave the relations invariant. (As the elements in question are generally merely the real elements, the correspondence may cease to be one to one if extended to the domain of imaginaries.) For him space is any one of these sets of elements, or if he uses the principle of abstraction it is the class of all such sets. Thus his postulates are both necessary and sufficient for the demonstration of all theorems in his space, and for him the complete story of his space is told. But for the psychologist and the metaphysician the complete story is not told in any so easy fashion. The psychologist is interested in tracing space concepts to their origin in sense experience, the metaphysician, in estimating the objective validity of these concepts. Whether the mathematician shall sometime be able to help them further than at present is difficult to say. Lie, as a geometer, distinguished between groups that were similar; and perhaps the logicians may some day get a sharper criterion than the one to one correspondence.

Force is the title of Part III, in which the subject of mechanics, in its widest significance, is treated. Poincaré avowedly passes over the difficulties of space, which he has just been treating, and of time, of which he had included no discussion in this book, and proceeds to difficulties essentially dynamical with the following characteristic preface (page 92): "Thus absolute space, absolute time, and even geometry are not conditions which impose themselves on mechanics; they are no more pre-

* BULLETIN, vol. 11 (1905), p. 405.

requisite to mechanics than the French language is logically prerequisite to the truths which are expressed in French. It would be possible to try to formulate the fundamental laws of mechanics in a language independent of all these conventions. Naturally the formulation of these laws would become more complicated, because all these conventions have been imagined precisely for the purpose of abbreviating and simplifying that formulation." It should seem, then, to use the author's guiding idea of convenience, that mechanics is rather a prerequisite to geometry than geometry to mechanics. Thus geometry does become in a certain large sense an experimental science, even if the experiments must be made upon objects which are not geometric but physical. (See the discussion by Bôcher.)

A little farther on in the text (page 100) is made a statement upon which the metaphysician may well meditate: "That a definition may be of value it is necessary that it shall teach us to *measure* force; this, moreover, is sufficient; it is by no means necessary that the definition teach us what force *as such* is, nor whether it be the cause or the effect of motion." Much misunderstanding and useless discussion between metaphysicians and scientists might be avoided if the former realized that what the latter needs to *know* about force is just what the careless baggage man needs to know about a trunk — its magnitude and its direction — and that what the real content of the thing is may better be left to the metaphysician himself, as a sometimes too officious customs officer, to examine. (The same remark probably holds equally well with regard to the insertion of psychological elements among our geometric axioms.) But who can resist the temptation to examine more critically than is absolutely necessary for the mere physicist? The author himself goes further and comes to the conclusion, similar to that for geometry, that the principles of mechanics are but conventions or disguised definitions, quite arbitrary except in so far as suggested by experience as the most convenient to adopt. It is during this discussion that the now famous formulation of the principle of the conservation of energy occurs, namely: "There is something which remains constant." And another statement: "The two propositions, 'the earth turns' and 'it is more convenient to suppose that the earth turns,' have one and the same meaning." And again, apropos of the law of acceleration: "If the acceleration of a visible body appears to depend on something else than the positions and velocities of other visible bodies and of the invisible molecules whose existence we have previously been brought to admit, nothing prevents us from supposing that this something else is the position or velocity of other molecules whose presence we have hitherto not suspected; and thus the law may be kept."

It requires these rather extensive quotations to bring clearly to light the author's underlying philosophical principle of convenience.* One might almost call his philosophy dogmatic. For, after obtaining a principle of wide application and becoming thoroughly used to interpreting nature or classifying experience by its aid, the most convenient thing to do is to introduce new terms and keep the principle rather than to find a new principle. Now undoubtedly for the mathematician such a procedure is always possible. For the physicist, however, it seems at times very artificial; and while it may serve as a first rate pragmatical doctrine to a certain recent school of metaphysics, there is apparently a decided tinge of superficiality about it. When one considers as practically identical the statements that the earth turns and that it is more convenient to imagine that the earth turns, he takes a position which seems to be either practical agnosticism or pure idealism, according to the interpretation one adopts. Whether this position is interpretable in such a way as to lead to a metaphysics satisfactory from all sides, only time can decide. Certainly, however, it is a great service that Poincaré has done in pointing out so clearly the fact that convenience plays a large rôle, that our knowledge is, as it were, a viscous fluid containing in itself traces of all knowledge that has been, with the weaker and more remote in time ever becoming less effective in determining the present state. The author seems to feel confident that we are approaching a steady state.

The last part of the work deals with nature. Before entering upon this subject the author draws a useful distinction between a law and a principle (page 141): "A law expresses a relation between two real terms A and B; but it is not rigorously true. We introduce arbitrarily an intermediate term C which is more or less fictive and which by *definition* is that which has to A *exactly* the relation expressed by the law. Thus our law is decomposed into an absolute and rigorous principle expressing the relation of A to C and an approximate experimental law expressing the relation of C to B." He goes on to say that in entering upon the study of nature we enter the

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* Compare Mach's well known principle of economy in thought.

domain of laws properly so called. From this it is evident that in studying mechanics it is principles which concern us, whereas in studying physics it is laws. It is difficult to see how any such finely drawn distinction can exist in view of the most recent developments of electrodynamic theory. It is becoming more and more convenient to consider electricity as fundamental and mass as derived therefrom. That Poincaré did not foresee this and take it into due account is no blot on his excellent little book; but it goes to show how careful one must be in asserting, for instance, that even the euclidean geometry will always remain the most convenient. Naturally euclidean geometry will always be taught, and so will newtonian mechanics; they will remain the first and simplest approximations to ultimate theories of space and moving matter. But when one is treating the fundamental question of the relation of hypothesis to science, it is not the first but the last available approximation which is of importance. The author would probably be the first to admit this.*

Although what precedes may go some of the way toward giving an impression of what the book contains, it cannot bring out the charm, frankness, and directness of the text. Fortunately one may take his choice of reading in the original French or in the excellent German translation.† If, however, the matter of language is of no very great importance to the reader, he should by all means choose the translation. For the translators have done more than give a faithful rendering of the text. They have written a preface which sets forth clearly the essential points in the argument, and have appended about a hundred pages of notes of great mathematical, historical, and bibliographical value to the reader. Finally they have added an exhaustive index. Why the French so persistently leave the indexes out of their books is hard to imagine. The practice is certainly a great inconvenience, especially in such a work as the present, where constant reference to what has gone before and to what is to follow is necessary for the proper appreciation of the different sections.

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
November, 1905

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* In fact the preface to his most recent series of essays "La valeur de la science," Paris, 1905, contains the statement that the changes in science which have taken place in the last few years call for a revision of some of the ideas propounded in the book here under review.

† An English rendering, containing a preface by Larmor and published by the Walter Scott Publishing Company, is reviewed by Russell in *Mind*, July, 1905, pp. 412–417.