

7) Axioms of completeness.—No motion is impossible unless it contradicts the above axioms.

From these are deduced the notions of surface, curve, and point. The places which can be occupied by the same solid, its surface or curve are termed congruent. The sphere, circle, straight line, plane, and angle are deduced from the notion of distance, which is a relation of all congruent couples of points. The so-called parallel postulate is deduced by proving that a certain continuous motion of a figure termed "immaterial quadrilateral" cannot contradict the above axioms.

F. N. COLE,
Secretary.

THE APRIL MEETING OF THE CHICAGO SECTION.

THE nineteenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Ill., on Saturday, April 14, 1906. The total attendance was forty, including the following members of the Society :

Dr. L. D. Ames, Mr. G. D. Birkhoff, Professor D. R. Curtiss, Professor E. W. Davis, Professor L. E. Dickson, Dr. E. L. Dodd, Mr. E. B. Escott, Dr. Peter Field, Professor G. W. Greenwood, Professor A. G. Hall, Professor E. R. Hedrick, Professor T. F. Holgate, Mr. Louis Ingold, Professor O. D. Kellogg, Dr. H. G. Keppel, Mr. N. J. Lennes, Professor H. Maschke, Professor E. H. Moore, Professor F. R. Moulton, Dr. L. T. Neikirk, Professor H. L. Rietz, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor H. E. Slaughter, Professor E. J. Townsend, Dr. W. D. Westfall, Professor D. T. Wilson, Mr. R. E. Wilson, Professor Alexander Ziwet.

The chairman of the Section, Professor Alexander Ziwet, presided at the two sessions. Owing to the large number of papers on the programme, it was voted to reduce the time allotments by twenty-five per cent. It was also voted hereafter to restrict the printed programme to those papers for which titles and abstracts are in the hands of the Secretary on the date specified in the preliminary call for the meeting, and to request this notice to be made in the BULLETIN in connection with the announcement of meetings of the Section.

A report of progress was made by the Executive Committee concerning the proposal to devote one or more sessions of the December, 1906, meeting to the consideration of mathematics for engineers, and to invite representatives of engineering schools to present papers and to take part in the discussion.

A resolution was introduced by Professor E. H. Moore and unanimously carried, expressing the very earnest hope of the Chicago Section that it may be found possible to establish a strong section of the Society which shall hold meetings at some convenient center in the Southwest.

The following papers were read :

- (1) Professor J. B. SHAW : "Significance of the term hyper-complex number."
- (2) Professor J. B. SHAW : "Mathematical processes."
- (3) Mr. LOUIS INGOLD : "A theorem on the intersections of Pascal lines."
- (4) Professor H. F. BLICHFELDT : "On modular groups isomorphic with a given linear group."
- (5) Professor H. F. BLICHFELDT : "On the order of linear homogeneous groups" (supplementary paper).
- (6) Mr. A. R. SCHWEITZER : "On the foundations of abstract geometry" (preliminary report).
- (7) Professor M. E. GRABER : "On the mathematical character of space intuition."
- (8) Dr. J. C. MOREHEAD : "Note on the factors of Fermat's numbers."
- (9) Mr. N. J. LENNES : "On functions of limited variation."
- (10) Professor E. R. HEDRICK : "On the function $\xi(h)$ in the law of the mean."
- (11) Professor F. R. MOULTON : "On the classes of periodic orbits computed by Professor G. H. Darwin."
- (12) Mr. W. D. MACMILLAN : "On a certain type of periodic orbits in the problem of three bodies."
- (13) Mr. F. L. GRIFFIN : "Certain periodic orbits of n finite bodies revolving about a relatively large central mass."
- (14) Mr. W. R. LONGLEY : "A class of periodic orbits of an infinitesimal body subject to the attraction of n finite bodies."
- (15) Professor H. L. RIETZ : "On normal correlation surfaces."
- (16) Professor L. E. DICKSON : "Linear algebras in which division is always uniquely possible."
- (17) Dr. E. L. DODD : "An application of Gibbs's exponential

dyadic function e^ϕ to the solution of linear vector and dyadic differential equations."

(18) Dr. E. L. DODD: "An example preparatory to the study of the ratio test for infinite series."

(19) Mr. R. E. WILSON: "On integral equations of the second kind."

(20) Mr. R. E. WILSON: "On integral equations of the first kind."

(21) Professor D. R. CURTISS: "A proof of the theorem concerning artificial singularities."

(22) Professor D. R. CURTISS: "On certain properties of wronskians and related matrices."

(23) Professor HEINRICH MASCHKE: "On spherical images of rectilinear congruences" (preliminary report).

(24) Dr. W. D. WESTFALL: "On integral equations of the second kind."

(25) Professor O. D. KELLOGG: "The roots of transcendental integral functions."

The papers of Professor Blichfeldt, Professor Graber, and Professor Maschke were read by title. Mr. MacMillan, Mr. Griffin, and Mr. Longley were introduced by Professor F. R. Moulton. Abstracts of the papers follow below, the numbers corresponding to the titles in the list above.

1. Professor Shaw's first paper compares critically the definitions of a hypercomplex number from the four standpoints of multiplex, n -dimensional number, operator, and extension of ordinary number. The first definition is exemplified in the definition of the complex number $a + b\sqrt{-1}$ as a couple (a, b) , where a and b are positive or negative reals, subject to the law of multiplication $(a, b)(c, d) = (ac - bd, ad + bc)$. The second is exemplified in the definition of $a + b\sqrt{-1}$ as a two dimensional number $ae_0 + be_1$, where $e_0^2 = e_0$, $e_1^2 = -e_0$, $e_0e_1 = e_1e_0 = e_1$. The third is exemplified in the definition of $a + b\sqrt{-1}$ as an operator which rotates through an angle $\tan^{-1} b/a$ and multiplies by $\sqrt{a^2 + b^2}$. The fourth defines $a + b\sqrt{-1}$ as a value of x which satisfies the ordinary equation $x^2 - 2ax + d = 0$, where $d = a^2 + b^2$. The latter algebraic definition is considered to be the most general since it does not require that from the equality of two hypercomplex numbers $a = b$ we must conclude $a_i = b_i$ ($i = 1, \dots, n$), where a_i is the i -th coordinate

of the number. This view permits us to call, for example, the algebra of positives and negatives a double or two-unit (qualitative unit) algebra, which the other definitions do not permit.

2. In his second paper Professor Shaw defined a mathematical process to be a definite procedure by which from given data we pass to a certain conclusion. The question of validity, objective or subjective, of the process is handed over to philosophy. The theory of mathematical processes bears to logic the relation that modern rational geometry has to euclidean geometry. It is also a generalized theory of mathematical forms. For example, on one hand the syllogism is exhibited as one of an infinite variety of modes of reasoning; on the other hand the theory of laws of combination, like the commutative law, is reduced to a problem in isomorphisms of substitution groups. This theory appears to be the common foundation of mathematics and logic. This paper is to be presented to the *Transactions* for publication.

3. In Mr. Ingold's paper are considered some properties of the configuration $\left(\begin{smallmatrix} 12 & 3 \\ 4 & 16 \end{smallmatrix} \right)$ in their relation to a conic through six of its points. From these properties are immediately deducible a large number of results as to collinearity and concurrence of points and lines connected with the inscribed hexagon and inscribed (any) pentagon. In particular it is shown that certain joins (other than Pascal lines) of diagonal points of the hexagon meet by sixes in sixty points which lie by fours on the sides of the hexagon and by threes on twenty other lines. The results applied to a reciprocal situation show that certain of the intersections of Pascal lines (Cayley's 360 points, *Quarterly Journal*, volume 9) lie by sixes on sixty lines which pass by tens through the vertices of the hexagon.

4. In his first paper Professor Blichfeldt proves the following theorem: Given a group G of linear homogeneous substitutions in n variables, transitive (irreducible) and of finite order, then there exists an infinitude of prime numbers p for which we can construct an isomorphic transitive group G' of linear homogeneous substitutions in n variables, the elements of whose matrices are integers taken modulo p .

5. Professor Blichfeldt's paper on the order of linear homogeneous groups is supplementary to his two former papers in

the *Transactions*. He reduces his former superior limit $(n - 1)(2n + 1)$ of the value of a prime p which may divide the order of a finite primitive group of linear homogeneous substitutions of determinant unity in n variables. If such a group has a substitution of order p ($p > n$) and variety m ($m \equiv n$), but none of order pk , then $p \equiv 3m + n - 3$. The primes which may divide the orders of the primitive collineation groups in 4, 5 and 6 variables are not greater than 11, 13 and 19, respectively. The two papers by Professor Blichfeldt will appear in the July number of the *Transactions*.

6. Mr. Schweitzer sets up systems of axioms for n dimensions ($n = 1, 2, 3, \dots$). Two undefined symbols enter the theory for n dimensions, viz., points as elements and a relation K , effective under conditions specified in the postulates, between two ordered n -ads of points. The postulates are laid down so as to permit of ready definition of betweenness, collinearity, etc. Systems III, etc., are shown to be sufficient for projective geometry. System I is easily extensible to higher dimensions on comparison with systems II, III, \dots . The extensions of the remaining systems are quite obvious. Such extensions are, however, less elegant than the corresponding sets in the first systems.

7. In this paper Professor Graber develops the theory that space intuitions do not change in nature, but in number and complexity. That in the naïve intuition of space the intuitional elements are few and their complexes restricted within narrow limits, but as geometric knowledge increases, the number of intuitions and intuitional complexes increases, and we have consequently more exact knowledge of spatial relations.

In the intuitional complexes formed by combinations of intuitions a, b, n , viz., $\Sigma(a, b, c, \dots, n)$, we may have (1) annihilators, e. g., $\widehat{ab} \equiv 0$; (2) correctors, e. g., $\widehat{ab} = a, b$, or a complex partaking of the nature of a and b ; (3) transmutants, e. g., $\widehat{ab} =$ a complex indicating a distinct advance.

It is then attempted to show that the ensemble of our space intuitions sustains a functional relationship to geometric progress, the intuitions leading to geometric research and vice versa. From this it would follow that no geometric problem is completely solved until we are able to interpret both the method of solution and the results in terms of the intuition.

8. Dr. Morehead's paper appears in full in the present number of the BULLETIN.

9. The variation of a function $f(x)$ on an interval ab is denoted by $V_a^b f(x)$. Mr. Lennes's paper is concerned with the variation of the definite integral regarded as a function of its limits of integration. It is known that $\int_a^x f(x)dx$ is a function of limited variation on an interval ab on which $f(x)$ is properly integrable. It is shown that

$$V_a^b \int_a^x f(x)dx = V_a^b \int_a^x |f(x)| dx.$$

In case the integral $\int_a^b f(x)dx$ exists improperly it is shown that the existence of $\int_a^b |f(x)| dx$ is a necessary and sufficient condition that $\int_a^x f(x)dx$ shall be a function of limited variation on ab . A new proof is given of a theorem proved by Jordan and Stolz, and announced by Moore with reference to these authors (*Transactions*, volume 2, page 322), viz., that the existence of the (Moore's) broad integral is sufficient for the existence of $\int_a^b |f(x)| dx$ either narrowly or broadly.

10. In this paper Professor Hedrick develops some of the properties of the function $\xi(h)$ in the law of the mean, written in the form

$$\frac{f(a+h) - f(a)}{h} = f'(a + \xi).$$

In particular it is shown that this law holds in a slightly modified form even in case the derivative $f'(x)$ exists merely with regard to certain assemblages. The possible discontinuities of $\xi(h)$ and of $f'(x)$ are then discussed. It is shown that $\xi(h)$ is always continuous at $h = 0$ with respect to an assemblage which has the power of the continuum in any interval about $h = 0$; and again that $f'(x)$ is not necessarily continuous for the entire assemblage of even those values which ξ actually assumes.

Finally, remarks concerning the general nature of a derivative, and concerning the necessary and sufficient conditions for a minimum of $f(x)$ close the paper. The paper will be presented to the *Annals of Mathematics*.

11. The paper of Professor Moulton consists of an application of the analysis which he presented to the Society at the summer meeting, 1905, to the periodic orbits discussed by Professor G. H. Darwin in *Acta Mathematica*, volume 21. The satellites of class A are of the type of the orbit of the moon when its inclination and eccentricity are neglected. A numerical example agreeing with Darwin's figures is given. The orbits of classes B and C are the analytic continuations of orbits whose coordinates are conjugate complex numbers for vanishing values of the disturbance. When expanded as power series in the lunar theory parameter m , the expressions for the coordinates of the orbits A , B and C all have the same radius of convergence. These series cease to converge for the value of m for which B and C become real; and this value of m probably gives their true radius of convergence. Numerical application is also made to planets A . The oscillating satellites were treated in an earlier paper.

12. Mr. MacMillan treats in his paper a certain type of periodic orbits of an infinitesimal body under the attraction of two finite bodies which revolve about one another in ellipses. The existence of periodic orbits of this type in the plane is demonstrated and a convenient method is given for constructing the solutions as power series in a parameter with periodic coefficients.

13. In this paper Mr. Griffin discusses a certain class of periodic solutions of the problem of n bodies where one of the masses is very large with respect to the others. In other words the distribution of the masses is such as is presented by the sun and any number of planets, or by a planet and any number of satellites. The bodies move in a plane, with their undisturbed synodic mean motions commensurable. Using the differential equations in all their generality, the existence of periodic orbits under the mutual disturbances is proved, the demonstration being particularly simple when all the bodies start from a "symmetric conjunction." A method of constructing the solutions as power series in the ratio of one of the

small masses to the large one is given in detail, the solution being unique when the period, the epoch, and the longitude of the initial conjunction have been selected. As an application, numerical work is given for the case of Jupiter's satellites, I, II, III. The fact that under certain circumstances there are no periodic orbits seems to offer information as to the reason for the existence of the "lacunary spaces" in the system of asteroids, and in Saturn's rings.

14. Mr. Longley assumes that the coordinates of the n finite bodies are known periodic functions of the time; for example, the generalized lagrangian solutions. The orbits considered are those of a particle of infinitesimal mass which revolves about one of the finite bodies. The problem is restricted to one plane, and the differential equations of motion are treated rigorously, the solutions being expressed as power series in parameters which converge when they are sufficiently small. It is shown that when a period (necessarily a multiple of the period of the finite bodies) is preassigned, there exist two and only two orbits of the particle, of the type considered, which are periodic with the prescribed period. The motion in one is direct and in the other retrograde. A convenient method for constructing the solutions is given, and some special cases are considered in detail.

15. Professor Rietz gives a derivation of correlation surfaces based upon the nature of certain plane sections of the surface. In the derivations hitherto given, the form of the function which defines the surface has either been assumed, or justified by assumptions as to the distribution and independence of the causes which produce deviations from the mean value. Then the constants in this function are determined. Much use has been made of correlation surfaces in recent years for the purpose of giving mathematical expression to facts of organic evolution. From the manner in which the applications are made, the nature of plane sections parallel to the two coordinate planes is of primary importance. It therefore seems well to make this the starting point of a derivation of these surfaces. It is shown in this paper how correlation surfaces which describe many natural phenomena can be obtained from certain types of sections.

16. Professor Dickson considers algebras whose elements are

$\Sigma a_i e_i$, the m coordinates a_i ranging over an arbitrarily given field F . For $m = 3$, he considers the most general linear transformation of the units and exhibits families of algebras invariant under every linear transformation. The algebras which admit more than one transformation (and hence exactly three transformations) into themselves are completely determined. From each of these standpoints he is led to the same remarkable set of families of algebras, each set characterized by a parameter μ . For $\mu = 1$, the family is the system of all fields of rank 3 with respect to F . For $\mu = 0$, the family is the system of algebras

$$i^2 = j, \quad ij = ji = b + \beta i + Bj, \quad j^2 = 4bB - \beta^2 - 8bi - 2\beta j,$$

where b, β, B range over the marks of F (assumed not to have modulus 2) for which $x^3 - b - \beta x - Bx^2$ is irreducible; in such an algebra division is always uniquely possible. For m an even integer ≥ 4 and F an arbitrary field, there exists a very interesting type of algebra in which division is always possible. For $m = 4$, the algebra is given by

$$i^2 = j, \quad ij = ji = k, \quad ik = ki = c, \quad j^2 = -c + dj, \quad \left(\begin{array}{l} c \text{ and } d^2 - 4c \\ \text{(not squares in } F) \end{array} \right) \\ jk = kj = -ci + dk, \quad k^2 = cd - cj$$

In fact, its determinant equals

$$(x^2 + dxy + cy^2)^2 - c(z^2 + dzw + cw^2)^2,$$

which vanishes only when x, y, z, w all vanish.

While there exists a large number of non-equivalent m -tuple algebras in a given field (for example, if $m = 3$ and F is the field of order 3 or 5, the number is 6 or 38 respectively), there appears to be a single non-field commutative algebra for each field when $m = 3$ or 4. However, for $m = 6$ there exist two non-equivalent, non-field commutative algebras in every field.

A general investigation was made of linear associative algebras in which division is always uniquely possible. An interesting example of such an algebra is one with the n^2 units $i_0^r e^s$ ($r, s = 0, 1, \dots, n - 1$), where $e^n = \tau =$ rational function of i_0 , and $e^s i_0^r = i_0^r e^s$, while $i_0, i_1 = \theta(i_0), i_2 = \theta^2(i_0), \dots$ are the roots of a uniserial abelian equation of degree n irreducible in F . Further, the coefficients of the latter and τ must satisfy certain conditions which can be expressed in terms of the theory of homogeneous forms of degree n . The paper will appear in the *Transactions*.

17. Dr. Dodd's first paper is in abstract as follows: Gibbs defines the direct product $\alpha \cdot \beta$ of two vectors as the product of their lengths into the cosine of the included angle. A dyad $\gamma\delta$ is an operator transforming one vector into another. Thus $(\gamma\delta) \cdot \alpha = (\delta \cdot \alpha)\gamma$. The product of two dyads is a dyad, *e. g.*, $(\gamma\delta) \cdot (\epsilon\zeta) = (\delta \cdot \epsilon)(\gamma\zeta)$. A dyadic, Φ , is a sum of dyads, and may be expressed in nonion form, thus:

$$\Phi = a_{11}ii + a_{12}ij + a_{13}ik + a_{21}ji + a_{22}jj + a_{23}jk + a_{31}ki + a_{32}kj + a_{33}kk,$$

where i, j, k , are unit vectors at right angles, and the a 's are scalars. In the product, $\Phi \cdot \Psi$, multiplication is distributive and, if Φ and Ψ are in nonion form, follows the row-column rule for the multiplication of determinants. $\Phi^2 \equiv \Phi \cdot \Phi$. The idem-factor $I = ii + jj + kk$ is such that $I \cdot \rho = \rho$ and $\rho \cdot I = \rho$, where ρ is the vector. Finally Gibbs defines

$$e^\Phi = I + \Phi + \frac{1}{2!}\Phi^2 + \frac{1}{3!}\Phi^3 + \dots$$

We find that, in general, $e^{\Phi+\Psi} \neq e^\Phi \cdot e^\Psi$, but if t and Δt are scalars, *i. e.*, numbers, real or complex, then $e^{(t+\Delta t)\Phi} = e^{t\Phi} \cdot e^{\Delta t\Phi}$; so that

$$\frac{d}{dt}e^{t\Phi} = \Phi \cdot e^{t\Phi} = e^{t\Phi} \cdot \Phi.$$

Then, if ρ is a vector and $\Gamma_0, \Gamma_1, \dots, \Gamma_n$ constant dyadics, a solution of

$$(1) \quad \Gamma_n \cdot \frac{d^n \rho}{dt^n} + \Gamma_{n-1} \cdot \frac{d^{n-1} \rho}{dt^{n-1}} + \dots + \Gamma_1 \cdot \frac{d\rho}{dt} + \Gamma_0 \cdot \rho = 0$$

will be $\rho = e^{t\Phi} \cdot \alpha$, if α is an arbitrary vector, and Φ is a solution of

$$\Gamma_n \cdot \Phi^n + \Gamma_{n-1} \cdot \Phi^{n-1} + \dots + \Gamma_1 \Phi + \Gamma_0 = 0.$$

In (1), ρ may be replaced by a dyadic Ψ if a dyadic Ω replaces α . If in (1) $\Gamma_n = c_n I$, etc., where the c 's are scalar constants, then (1) reduces to

$$c_n \frac{d^n \rho}{dt^n} + c_{n-1} \frac{d^{n-1} \rho}{dt^{n-1}} + \cdots + c_1 \frac{d\rho}{dt} + c_0 =$$

Here substitute $\rho = e^{mt} \cdot \alpha = e^{mt} \alpha$, where m is a scalar. Cf. Grassmann's Ausdehnungslehre, § 498.

18. The example of ratio test devised by Dr. Dodd is the following :

$$\begin{aligned} \text{Let} \quad S &= u_1 + u_2 + u_3 + \cdots + u_n + \cdots \\ &= \frac{1}{5} + \frac{1 \cdot 3}{5 \cdot 10} + \frac{1 \cdot 3 \cdot 5}{5 \cdot 10 \cdot 15} + \cdots + \frac{1 \cdot 3 \cdots (2n-1)}{5 \cdot 10 \cdots 5n} + \cdots \end{aligned}$$

$$\text{Then} \quad r = \frac{u_{n+1}}{u_n} = \frac{2n+1}{5(n+1)}, \quad \lim_{n \rightarrow \infty} r = \frac{2}{5}, \quad r < \frac{2}{5}.$$

The sum of S exists, and is less than 0.2912, by comparison with

$$\begin{aligned} S' &= u_1 + u_2 + u_3 + u_4 + u_5 + (2/5)u_5 + (2/5)^2 u_5 \\ &\quad + (2/5)^3 u_5 + \cdots \end{aligned}$$

The sum of S is greater than .2909, by comparison with

$$S'' = u_1 + u_2 + u_3 + u_4 + u_5 + (11/30)u_5 + (11/30)^2 u_5 + \cdots$$

Then $S = 0.291$ is correct to within 0.0002.

An infinite series whose test ratio approaches a limit l will ultimately become approximately a geometric series with ratio l , and it will converge if $|l| < 1$.

19. Mr. Wilson's first paper treats of the reduction of a differential equation of the third order under certain boundary conditions to an integral equation of the second kind, and a generalization to differential equations of order $2n - 1$. After showing that the matrix of such an integral equation is always skew-symmetric, he applies his results to the differential equation $y''' - 4py' - 2p'y = 0$, which arises in the solution of Lamé's equation $y'' = (Ax + B)y$ by the application of a method due to Hermite, and shows that for a fixed value of A there exists an infinity of normal solutions which satisfy the given boundary

conditions. He further shows under what conditions an arbitrary function is developable in terms of these normal solutions.

20. In his second paper Mr. Wilson discusses the representation of arbitrary functions by means of Fourier's double integrals and shows the relation of the theory of integral equations of the first kind to the theory of Fourier's integrals. Incidentally he gives a few examples of integral equations of the first kind which are readily solved and which in themselves might serve as the basis of Fourier's theory.

21. The first paper of Professor Curtiss gives a proof of the fundamental theorem which states that if $f(z)$ is single valued and analytic at all points in the neighborhood of a point c , exclusive of c , and remains finite throughout this neighborhood, it has at most an artificial singularity at c ; *i. e.*, with a suitable definition of $f(c)$ the function $f(z)$ will be analytic at the point c . In some fallacious proofs use has been made of the auxiliary function

$$\phi(z) = (z - c)f(z) \quad (z \neq c), \quad \phi(c) = 0.$$

A very simple demonstration is here given, based on the consideration of the function $(z - c)\phi(z)$. This paper will be presented to the *Annals of Mathematics* for publication.

22. The second paper of Professor Curtiss will appear in the July number of the BULLETIN.

23. Professor Maschke discussed the solution of the general problem of the determination of rectilinear congruences whose spherical images are given, *i. e.*, the determination of x, y, z in terms of X, Y, Z and the quantities e, f, f', g defined by

$$\Sigma \frac{\partial X}{\partial u} \frac{\partial x}{\partial u} = e, \quad \Sigma \frac{\partial X}{\partial u} \frac{\partial x}{\partial v} = f, \quad \Sigma \frac{\partial X}{\partial v} \frac{\partial x}{\partial u} = f', \quad \Sigma \frac{\partial X}{\partial v} \frac{\partial x}{\partial v} = g.$$

24. In this paper Dr. Westfall shows that the restriction that the matrix $K(s, t)$ be symmetric in s, t in Hilbert's treatment of the integral equation of the second kind

$$f(s) = \phi(s) - \lambda \int_0^1 K(s, t)\phi(t)dt$$

is not necessary for that treatment. A slight change in his proof gives the relations

$$K(s, t) + K(s, t) = \lambda \int_0^1 K(s, r)K(r, t)dr,$$

$$K(s, t) + K(s, t) = \lambda \int_0^1 K(s, r)K(r, t)dr,$$

which prove the existence and uniqueness of the solution

$$\phi(s) = f(s) - \lambda \int_0^1 K(s, t)f(t)dt.$$

25. The fact that the roots of an integral algebraic function are continuous functions of the coefficients may be generalized to transcendental functions, and the result very simply applied to give certain information concerning the roots of the latter. Professor Kellogg proposes two applications of this notion, the first in building up a transcendental integral function term by term, so that it appears that if the convergence of the series is rapid enough, it will surely have finite roots. By a second application the series is considered as a polynomial plus a remainder. If the remainder is sufficiently small all the roots of the polynomial have corresponding roots in the complete function.

H. E. SLAUGHT,
Secretary of the Section.

CHICAGO, ILL.,
April 20, 1906.

GROUPS IN WHICH ALL THE OPERATORS ARE
CONTAINED IN A SERIES OF SUBGROUPS
SUCH THAT ANY TWO HAVE ONLY
IDENTITY IN COMMON.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, April 28, 1906.)

1. We begin with the case where the group G is any abelian group such that all of its operators are contained in a series of subgroups $H_1, H_2, \dots, H_\lambda$ any two of which have only