# THE CONSTRUCTION OF A FIELD OF EXTREMALS ABOUT A GIVEN POINT.

BY PROFESSOR G. A. BLISS.

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In establishing the sufficiency of the well-known conditions for the simplest problem of the calculus of variations in the plane, an essential step is the proof that a one-parameter family of extremals simply cover a portion of the plane and constitutes a so-called field. It has been shown by several writers * that when all the extremals of the family pass through a single point $O$, any arc of one of them can be imbedded in a field provided that it does not contain the point $O$ or its conjugate. But it is also true that the field extends up to and surrounds $O$, as will be proved in the following paragraphs.†

When the point $O$ is not a singular point of the problem it may, without loss of generality, be taken as the origin, and the equations of the extremals through it can always be found in the form

$$x = \phi(s, t), \quad y = \psi(s, t). \tag{1}$$

The parameter $s$ is the length of arc measured from $O$, and $t$ is the angle between the $x$-axis and the direction of the extremal at $O$. The functions $\phi$ and $\psi$ therefore satisfy the identities

$$\begin{gathered} \phi_s^2 + \psi_s^2 \equiv 1, \quad 0 \equiv \phi(0, t), \quad 0 \equiv \psi(0, t), \\ \cos t \equiv \phi_s(0, t), \quad \sin t \equiv \psi_s(0, t), \end{gathered} \tag{2}$$

where the subscripts denote partial differentiation. They are furthermore supposed to be of class $C'$‡ for

$$0 \leqq s \leqq S, \quad t = \text{any value},$$

and, since one and only one extremal passes through $O$ in a given direction, to be periodic with period $2\pi$ in $t$.

It is proposed to show that a circle of radius $\rho$ can be found

* Osgood, *Annals of Mathematics*, 2d ser., vol. 2 (1901), p. 112. Bolza, Lectures on the calculus of variations (1904), pp. 78, 175. Goursat, Cours d'analyse, vol. 2, p. 613.

† In his lectures on the calculus of variations, summer semester, 1879, Weierstrass stated that a field including the point $O$ could be constructed, but omitted the proof.

‡ *I. e.*, continuous with continuous first partial derivatives.

with center at $O$ such that the equations

$$(3) \qquad r \cos \alpha = \phi(s, t), \quad r \sin \alpha = \psi(s, t)$$

have one and only one solution $(s, t)$ corresponding to any pair of values $(r, \alpha)$ for which

$$(4) \qquad 0 < r < \rho, \quad \alpha = \text{any value},$$

and which correspond therefore to a point in the circle. The pair of functions

$$(5) \qquad s = s(r, \alpha), \quad t = t(r, \alpha)$$

so defined turn out to be of class $C'$ in the region (4) and periodic in $\alpha$ with period $2\pi$.

First consider $r$ as a function of $s$ and $t$,

$$r^2 = \phi^2 + \psi^2.$$

The value of each of the expressions

$$(6) \qquad \frac{dr}{ds}, \quad \frac{\phi_s \psi_t - \phi_t \psi_s}{\phi\phi_s + \psi\psi_s}$$

at the point $O$ $(s = 0)$ is unity. The derivative $dr/ds$ has the value

$$(7) \qquad \frac{dr}{ds} = \frac{\phi\phi_s + \psi\psi_s}{\sqrt{\phi^2 + \psi^2}},$$

and by Taylor's formula

$$\phi(s, t) = s \cdot \phi_s(\theta s, t), \quad \psi(s, t) = s\psi_s(\theta' s, t).$$

By substituting these values in (7) and letting $s$ approach zero the desired result for $dr/ds$ is derived. If $\Delta(s, t)$ denotes the numerator of the second of the expressions (6), the value of $\Delta_s$ is

$$\Delta_s(s, t) = \phi_s \psi_{ts} - \phi_{ts} \psi_s + \phi_{ss} \psi_t - \phi_t \psi_{ss}.$$

At the point 0

$$\Delta_s(0, t) = 1$$

on account of the equations (2), from which may be derived also

$$\phi_{ts}(0, t) = -\sin t, \quad \psi_{ts}(0, t) = \cos t.$$

It follows then that the fraction

$$\frac{\phi_s\psi_t - \phi_t\psi_s}{\phi\phi_s + \psi\psi_s} = \frac{s\Delta_s(\theta''s, t)}{s\phi_s(\theta s, t)\,\phi_s + s\psi_s(\theta's, t)\,\psi_s}$$

takes on the value unity when $s$ approaches zero.

Since the expressions (6) are equal to unity at the point $O$ ($s = 0$), there will exist a largest value $S_1$ such that the inequalities

$$\frac{dr}{ds} > 0, \quad 2 > \frac{\phi_s\psi_t - \phi_t\psi_s}{\phi\phi_s + \psi\psi_s} > 0 \tag{8}$$

hold for all points $(s, t)$ in the region

$$0 < s < S_1 (\leqq S), \quad t = \text{any value.} \tag{9}$$

If $\rho$ denotes the minimum value of $r(S_1, t)$, the circle of radius $\rho$ about the point $O$ will be simply covered by the extremals (1) and will be a circle, therefore, of the kind originally sought.

To prove this, consider any circle about $O$ of radius $r_1 < \rho$. It is cut once and only once by each of the extremals (1), and consequently the equation

$$r_1^2 = \phi^2 + \psi^2$$

has one solution $s$ in the interval (9) corresponding to each value of $t$. By the well-known theorems on implicit functions the function $s(t)$ so defined is of class $C'$, since the derivative $\phi\phi_s + \psi\psi_s$ can not vanish (see (7) and (8)). When $r = r_1$ and this function $s(t)$ are substituted in the equations (3), $\alpha$ is determined as a function of $t$ whose derivative by an easy calculation is found to be

$$\frac{d\alpha}{dt} = \frac{\phi_s\psi_t - \phi_t\psi_s}{\phi\phi_s + \psi\psi_s}.$$

From (8) it follows therefore that $\alpha$ always increases as $t$ increases from 0 to $2\pi$. Furthermore the value $\alpha(2\pi)$ differs from $\alpha(0)$ by some multiple of $2\pi$, since there is only one extremal through the point 0 in the direction $t = 0$, or what is the same thing, in the direction $t = 2\pi$. But from (8) again

$$\alpha(2\pi) < \alpha(0) + 2 \cdot 2\pi,$$

and one sees therefore that $\alpha(2\pi) = \alpha(0) + 2\pi$. It is evident that the two intervals $\alpha(0) \leqq \alpha \leqq \alpha(0) + 2\pi$ and $0 \leqq t \leqq 2\pi$ are in one-to-one correspondence.

From what precedes it follows that *to each value of $r$ and $\alpha$ satisfying the conditions*

$$(10) \qquad 0 < r < \rho, \quad \alpha = \textit{any value}$$

*there corresponds one and but one pair of values $s$, $t$ satisfying the conditions*

$$0 < s < S_1, \quad t = \textit{any value}$$

*and the equations*

$$(11) \qquad r \cos \alpha = \phi(s, t), \qquad r \sin \alpha = \psi(s, t).$$

*The implicit functions*

$$s = s(r, \alpha), \quad t = t(r, \alpha)$$

*so defined are of class $C'$ in the neighborhood of any point* $(r, \alpha)$ *of the region* (10), since by (8) the determinant $\Delta = \phi_s\psi_t - \phi_t\psi_s$ of equations (11) is always different from zero when $s$ and $t$ satisfy the inequalities (9). It is easily seen that any two values of $\alpha$ differing by a multiple of $2\pi$ define with $r$ the same values of $s$ and $t$.

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# SOME PARTICULAR SOLUTIONS IN THE PROBLEM OF $n$ BODIES.

BY DR. W. R. LONGLEY.

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LET the masses of $n$ finite bodies be represented by $m_1, m_2, \cdots, m_n$. Suppose that the bodies lie always in the same plane, and that their coordinates with respect to their common center of mass as origin and a system of rectangular axes which rotate with the uniform angular velocity $N$ are, respectively, $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. Supposing that the bodies attract each other according to the newtonian law, the differential equations of motion are

$$(1) \qquad \begin{aligned} \frac{d^2x_i}{dt^2} - 2N\frac{dy_i}{dt} - N^2x_i &= -\sum_{j=1}^{n}\frac{m_j(x_i - x_j)}{r_{i,j}^3}, \\ \frac{d^2y_i}{dt^2} + 2N\frac{dx_i}{dt} - N^2y_i &= -\sum_{j=1}^{n}\frac{m_j(y_i - y_j)}{r_{i,j}^3}, \end{aligned}$$