THE FOURTEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

Many members of the Society recall with pleasure the summer meeting and colloquium held at Ithaca in 1901. After an interval of six years the summer meeting, the fourteenth of the series, was again convened at Cornell University on Thursday and Friday, September 5–6, 1907. By close economy of time the scientific proceedings were condensed into two sessions on Thursday and a morning session on Friday. Friday afternoon was devoted to an excursion on Lake Cayuga, Mr. H. H. Westinghouse, of the University, having kindly placed his steam yacht at the disposal of the members. The evening gatherings at the Town and Gown Club also furnished pleasant opportunities for social intercourse.

The first session opened with an address of welcome by Professor Wait. At the close of the meeting resolutions were adopted expressing the Society's appreciation of the generous hospitality of the University and its officers.

The attendance included the following forty-seven members of the Society:

Professor O. P. Akers, Professor G. A. Bliss, Mr. W. C. Brenke, Professor W. G. Bullard, Dr. W. B. Carver, Professor F. N. Cole, Professor L. L. Conant, Professor E. W. Davis, Professor S. C. Davisson, Dr. E. L. Dodd, Dr. F. J. Dohmen, Professor W. P. Durfee, Professor L. P. Eisenhart, Professor H. B. Fine, Professor W. B. Fite, Professor W. B. Ford, Professor A. S. Gale, Professor C. N. Haskins, Professor E. R. Hedrick, Professor J. I. Hutchinson, Dr. L. C. Karpinski, Professor O. D. Kellogg, Professor T. E. McKinney, Professor James McMahon, Professor W. H. Metzler, Mr. E. A. Miller, Professor G. A. Miller, Dr. R. L. Moore, Mr. F. W. Owens, Dr. Arthur Ranum, Professor E. D. Roe, Professor D. A. Rothrock, Dr. F. R. Sharpe, Miss M. E. Sinclair, Dr. C. H. Sisam, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Professor J. H. Tanner, Professor Anna L. Van Benschoten, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor L. A. Wait, Dr. W. D. A. Westfall, Professor H. S. White, Professor J. M. Willard, Professor Alexander Ziwet.

The President of the Society, Professor H. S. White, occupied the chair, being relieved by Professors Fine and E. B. Van Vleck. The Council announced the election of the following persons to membership in the Society: Mr. Thomas Buck, University of Chicago; Mr. Arnold Dresden, University of Chicago; Mr. T. H. Hildebrandt, University of Chicago; Mr. W. J. King, Harvard University; Mr. J. O. Mahoney, High School, Dallas, Texas; Dr. J. F. Messick, Randolph-Macon College; Mr. H. W. Powell, College of the City of New York. Six applications for membership in the Society were received. The total membership is now 569.

The following papers were read at the summer meeting:

(1) Professor L. E. Dickson: "Modular theory of group matrices."

(2) Professor W. B. Ford: "Sur les équations linéaires aux différences finies."

(3) Professor R. D. Carmichael: "On the classification of plane algebraic curves possessing fourfold symmetry with respect to a point."

(4) Professor R. D. Carmichael: "Note on certain inverse problems in the simplex theory of numbers."

(5) Dr. W. B. Carver: "The ten special $\Gamma^4_{6,2}$ configurations in the Pascal hexagram."

(6) Professor E. O. Lovett: "Generalization of a problem of Bertrand in mechanics."

(7) Professor E. O. Lovett: "The invariants of a group which occurs in the problem of n bodies."

(8) Professor E. R. Hedrick: "A peculiar example in the theory of surfaces."

(9) Professor E. R. Hedrick: "A smooth closed curve composed of rectilinear segments."

(10) Professor R. D. Carmichael: "On certain transcendental functions defined by a symbolic equation."

(11) Dr. D. C. Gillespie: "On the canonical substitution in the Hamilton-Jacobi canonical system of differential equations."

(12) Professor G. A. Miller: "The invariant substitutions under a substitution group."

(13) Professor G. A. Miller: Methods of determining the primitive roots of a number."

(14) Professor Virgil Snyder: "On a special algebraic curve having a net of minimum adjoint curves."

(15) Professor JAMES MCMAHON: "The differential geometry of the vector field. Second paper: lamellar field."

(16) Professor L. E. DICKSON: "Commutative linear groups."

(17) Professor L. E. DICKSON: "A simple derivation of the canonical forms of linear transformations."

(18) Professor EDWARD KASNER: "Geometric interpretation of integrating factors."

(19) Professor EDWARD KASNER: "The conformal representation of geodesic circles."

(20) Mr. A. R. SCHWEITZER: "On the relation of right-handedness in geometry."

(21) Dr. F. L. GRIFFIN: "On the law of gravitation in the binary systems, II."

(22) Dr. F. L. GRIFFIN: "Certain trajectories common to different laws of central force."

(23) Professor E. W. DAVIS: "Colored imaginaries. I, Imaginaries in the plane."

(24) Professor E. W. DAVIS: "Colored imaginaries. II, Imaginaries in space."

(25) Dr. C. H. SISAM: "On the equations of quartic surfaces in terms of quadratic forms."

(26) Professor VIRGIL SNYDER: "On the range of birational transformation of curves having genus greater than that of the canonical form."

(27) Professor G. A. MILLER: "Third report on recent progress in the theory of groups of finite order."

(28) Professor OSWALD VEBLEN: "Continuous increasing functions of ordinal numbers."

(29) Professor H. S. WHITE and Miss K. G. MILLER: "Note on Lüroth's type of plane quartic curve."

(30) Professor W. B. FITE: "Concerning the degree of an irreducible linear homogeneous group."

(31) Dr. ARTHUR RANUM: "Concerning linear substitutions of finite period with rational coefficients."

(32) Professor R. P. STEPHENS: "Certain curves of class n having $n - 2$ tangents in any given direction."

(33) Professor ANNA L. VAN BENSCHOTEN: "Curves of genus 4 which remain invariant under birational transformation."

(34) Miss M. E. SINCLAIR: "On a discontinuous solution in the problem of the surface of revolution of minimum area."

(35) Dr. MAURICE FRÉCHET: "Sur les opérations linéaires (troisième note)."

(36) Professor A. G. GREENHILL: "The elliptic integral in electromagnetic theory."

Dr. Fréchet's paper was communicated to the Society through Professor E. B. Van Vleck. In the absence of the authors, Dr. Gillespie's paper was presented by Professor Snyder, Dr. Griffin's papers by Professor Veblen, Professor Greenhill's paper by Professor McMahon, and the papers of Professor Dickson, Professor Carmichael, Professor Lovett, Professor Kasner, Mr. Schweitzer, Professor Stephens and Dr. Fréchet were read by title.

Professor Dickson's first paper was published in the July number of the *Transactions*. Professor Miller's first paper appeared in the October BULLETIN. Professor Miller's report, Professor Snyder's first paper, and Professor Carmichael's second paper are published in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

2. Professor Ford's paper considers the linear difference equation

$$a_0(x)\Delta^n y + a_1(x)\Delta^{n-1}y + \cdots + a_n(x)y = 0,$$

and obtains certain results analogous to those established during recent years by Dini for the linear differential equation

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = 0.$$

Four general theorems are established concerning the existence and character of the solutions of the above difference equation, the last three of which are especially adapted to the study of the solutions $y(x)$ in the neighborhood of the point $x = \infty$. The results obtained find noteworthy application in the study of the convergence of algebraic continued fractions, as will appear more at length subsequently. The paper appeared in the last number (volume 13, number 4) of the *Annali di Matematica.*

3. Professor Carmichael considers the classification of plane algebraic curves possessing fourfold symmetry with respect to a point. The discussion is a generalization of his previous paper on quartic curves possessing the same property. It results in the following theorem:

Excluding curves consisting of isolated points only, the only loci of the nth order possessing fourfold symmetry with respect to a point taken as the origin are those whose rectangular cartesian equations are of one of the forms

$$\Sigma[a_{ts}(x^t y^s + (-1)^t x^s y^t)] = 0, \qquad \Sigma[a_{ts}(x^t y^s - (-1)^t x^s y^t)] = 0,$$

where n must be even and $s = 0, 1, 2, \cdots, n$; $t = s, s+1, \cdots, n$; subject to the condition that $t + s = 0, 2, 4, \cdots, n$.

The paper will be published in the *Annals of Mathematics*.

5. In his papers, "Sur quelques théorèmes, etc.," *Crelle's Journal*, volume 31 (1846), and "On Pascal's theorem," *Quarterly Journal*, volume 9 (1868), Cayley called attention to a peculiar special case of the configuration $\Gamma^4_{6,2}$ which occurs ten times in the Pascal hexagram. In a paper "On the Cayley-Veronese configurations," *Transactions*, volume 6 (1905), Dr. Carver showed that there is a pencil of six conics connected with any $\Gamma^4_{6,2}$. The pencils connected thus with the special $\Gamma^4_{6,2}$'s in the Pascal hexagram are studied in the present paper, and their relation to the fundamental conic of the hexagon is shown. The notation used is defined in the author's paper cited above.

6. In the plane xoy let a point (x, y) be acted upon by an acceleration force whose components are certain functions of the coordinates of the point. Professor Lovett proposes to examine in what cases all the trajectories, whatever be the initial conditions, are curves of the family defined by the differential equation

$$C(\alpha, \beta, \gamma, \delta, \epsilon) \equiv \alpha(12)^2(15) + \beta(12)^2(24) + \gamma(12)(13)(14) + \delta(13)^3 + \epsilon(12)(13)(23) = 0,$$

where α, β, γ, δ, ϵ are constants, and (i, j) is written for $x^{(i)}y^{(j)} - x^{(j)}y^{(i)}$, the indices indicating derivations with regard to any independent variable. If the left member is $C(9, 45, -45, 40, -90)$, we have the form in which Stéphanos has written the differential equation of the conics first given by Monge.

This problem was studied by Bertrand * in the case where

* Bertrand: "Note sur un problème de la mécanique," *Comptes rendus*, vol. 84, p. 731. Bertrand had formulated the problem in a previous note "Sur la possibilité de deduire d'une seule les lois de Kepler et le principe de l'attraction," ibid., p. 673.

the force has in general a unique direction at every point and the trajectories are conic sections. Bertrand showed that in this case the force ought either to pass through a fixed point or be parallel to a fixed direction. Stéphanos * has recently examined the general case of conic trajectories when the force has not necessarily a unique direction; he finds that the problem admits of no other solution than those included in the case considered by Bertrand, and shows that every force under which a point describes a conic section, whatever be the initial conditions, always passes through a fixed point or remains parallel to a fixed direction. Finally by applying his method to the determination of the explicit forms of the forces in the case of conics Stephanos rediscovers the known theorems of Darboux and Halphen.†

In the present note free use is made of the method and formulas developed by Stéphanos in his memoir to which reference has been made. The analytic conditions determining the forces are set up and transformed with a view to make appear the condition for central forces or forces parallel to a fixed direction; it results that this condition may be satisfied and one of the parameters in the above differential equation remain absolutely arbitrary, that is, if α, γ, δ are in the ratios $9 : -45 : 40$, and ϵ equals -2β, the forces under which a point describes the corresponding curve, whatever be the initial conditions, either pass through a fixed point or remain parallel to a fixed direction. Furthermore in this case, which obviously includes the conic sections, the differential equations to be integrated for the forces are identical in form to those encountered by Stéphanos; his results and those of Darboux and Halphen thus become applicable to this more general problem.

7. If in the problem of the motion of n bodies about a fixed center under forces varying as an arbitrary function of the distance we seek those integrals which do not involve the law of force,‡ we come upon a system of linear partial differential equations formed by equating to zero $n(2n+1)$ infinitesimal

* Cyparissos Stéphanos: "Sur les forces donnant lieu à des trajectories coniques," *Crelle's Journal*, vol. 131 (1906), pp. 136–151.

† Darboux, Halphen, *Comptes Rendus*, vol. 84, pp. 760, 939.

‡ Gravé seems to have been the first to seek integrals of this kind in the problem of three bodies, and he found that the differential equations in the Bour-Bertrand form admit of no integrals independent of the law of force other than those already known. See his paper: "Sur le problème des trois corps," *Nouv. Ann. de Math.*, series 3, vol. 15 (1896), pp. 537–547.

transformations. That these infinitesimal transformations generate a continuous group was pointed out by Professor Lovett in an unpublished paper previously presented to the Society (December 29, 1902); in the present paper the methods of Sophus Lie are employed to construct the invariants of this group.

8. In Professor Hedrick's first paper an example of a surface is given in which the distance from a point on the normal to each normal section is a minimum, while the distance from the same point to the surface itself is not a minimum. The example shows the fallacy of a geometric conclusion sometimes assumed as obvious, and reference is made to a recent use of the fallacious theorem. (BULLETIN, volume 13, No. 9, page 447.)

9. In this note, which will be offered for publication to the *Annals of Mathematics*, Professor Hedrick gives an example of a closed curve formed by the junction of (an infinite number of) rectilinear segments, which is nevertheless *smooth*, in the sense defined by Professor Osgood (Funktionentheorie). The statement is to be understood in the sense that between any two points on the curve there is an arc which is a segment of a straight line. The example is obtained by simple integration from a well-known case; it seems worth while in illustrating the non-intuitive character of the ideas involved.

10. In Professor Carmichael's third paper a large class of transcendental functions is defined in the following manner: Let $u = f(x)$ and $v = f(y)$, and put $u^v = v^u$. This last equation has a solution $u = f(x) = \mu^{1/(\mu-1)}$, $v = f(y) = \mu^{\mu/(\mu-1)}$, where μ is a variable. Solving these functional equations, we have $x = g(\mu^{1/(\mu-1)})$ and $y = g(\mu^{\mu/(\mu-1)})$, where g is the function inverse to f. These equations define a function y of x. It is this function which is studied in the paper. Attention is confined chiefly to questions of continuity and development in series.

11. Lie established a one to one correspondence between the integrals of the canonical system of differential equations and the one-parameter continuous groups of contact transformations of which the system admits: *i. e.*, making use of an integral,

one can construct the infinitesimal transformation belonging to a group of which the system admits, and conversely by making use of the infinitesimal transformation one can construct an integral of the system.* This theorem is the foundation of the modern transformation theorems of dynamics.†

The single canonical substitution on the other hand has served chiefly in the simplification of the equations.‡ The paper of Dr. Gillespie defines a single canonical substitution, the knowledge of which leads immediately to the construction of an integral of the system.§

13. Professor Miller's note aims to exhibit some elementary relations between well known methods of finding the primitive roots of a number and the properties of a cyclic group. It also contains a very elementary proof of the following theorem, which is believed to be new: When the exponent to which the primitive roots of a given odd number g belong is of the form $2q$, q being any odd prime, each of the primitive roots of g is given once and only once by $-\alpha^2$, $1 < \alpha < g/2$ and α being prime to g. When g is even, the primitive roots are obtained by replacing the even values of α^2 by $g/2 - \alpha^2$, while the odd values remain unchanged. In the former case α^2 belongs to exponent q, while α^2 or $g/2 + \alpha^2$ belong to this exponent for odd or even values of α, respectively, when g is even and has primitive roots. From this we derive the following useful corollary: Every prime of the form $2q + 1$, q being any odd prime, has for its primitive roots $-\alpha^2$, $1 < \alpha < q + 1$. It is also pointed out that the theorem which affirms that the product of all the primitive roots of any number which has more than one such root is congruent to unity is a special case of the theorem that the product of all the operators of the same order > 2 in any abelian group is identity.

15. Professor McMahon's paper considers the field of a vector whose three components at any point are the partial deriva-

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* Whittaker, Analytical dynamics, p. 308.

† l. c., p. 292.

‡ Jacobi, Vorlesungen über Dynamik. Poincaré, Mécanique céleste, vol. 1. For condition that a substitution be canonical, see Lie, *Norw. Ark. für Math.*, etc., 1877. Also Jacobi, Vorlesungen über Dynamik.

§ For the two theorems upon which the argument depends, see the author's thesis, "Anwendungen des Unabhängigkeitssatzes auf die Lösung der Differentialgleichungen der Variationsrechnung," p. 30; Göttingen, 1906. Also, "On the construction of an integral of Lagrange's equations in the calculus of variations," BULLETIN, vol. 13 (1907), pp. 345–348.

tives of a given function of the coordinates of the point. Among the topics treated are the following: Derivative of the vector for any direction in its field; variation of derivative; variation of tensor; rotation of the field vector for displacement in any direction; variation of rotation; lines of no rotation; family of surfaces orthogonal to the vector lines; relation of the indicatrix conic at any point to the vector field around the point; invariants of the field.

16. In the second paper by Professor Dickson it is shown at the outset that every maximum (*i. e.*, not contained in a larger) commutative group M of m-ary linear homogeneous transformations (a_{ij}) in a field F may be completely defined by linear homogeneous relations between the a_{ij} with constant coefficients belonging to F. In other words, the independent parameters of M enter linearly and homogeneously. Continuous commutative linear groups with the last property have been considered by Professor W. Burnside, *Proceedings of the London Mathematical Society*, volume 29 (1898), pages 325-352, chiefly from the standpoint of the invariant factors of the characteristic determinant of the general transformation of the group. He treats in detail the case of two invariant factors distinct from unity; but (page 341) draws an erroneous conclusion (that $b_1 = 0$, whereas $a_1 - e_1$ is zero in view of the initial hypothesis), which renders invalid the first step, as well as all the successive steps, of his reduction process. Moreover, of the groups thus omitted, there occur examples with $b_1 \neq 0$ which are not reducible to groups with $b_1 = 0$ and of the type considered. Burnside starts with a particular transformation S (in canonical form) with the desired invariant factors, then exhibits the most general linear transformation T commutative with S, and finally attempts to select and normalize all the transitive commutative groups composed exclusively of transformations T. He expected to prove that all such groups have the same invariant factors as the individual transformation S. It is clear *à priori* that this will be true only in trivial cases. A correct version of Burnside's methods of constructing commutative linear groups would therefore still have the objection of being highly redundant. To establish the non-equivalence of two groups, Burnside makes use (§11) of another classification (§9), based upon a highly desirable normal form of the commutative linear group. In this normal form, at least one variable x_i is merely

multiplied by a constant, at least one y_j is replaced by $by_j + \Sigma a_i x_i$, etc. As Burnside omits references, it should be said that this normal form was given implicitly by Lie in 1878 (cf. Lie-Engel, Transformationsgruppen, I, page 589), and explicitly for commutative linear groups of order a power of p in the $GF[p^\nu]$ by Jordan in 1870 (Traité des substitutions, page 147). However, the latter author makes errors of various kinds (on page 148) in determining the commutative groups of the simplest types, as is pointed out in detail in the present paper. By means of the Jordan normal form (established for a general field F) and the theorem stated at the outset, it is a comparatively simple algebraic problem to determine explicitly all the maximum commutative linear groups. The more difficult problem of their reduction to non-equivalent types is found to depend upon the normalization of families of quadratic forms $\Sigma c_i q_i$ by linear transformation both of the parameters c_i and the variables in q_i. The paper will be offered for publication in the *Annals of Mathematics*.

17. The third paper by Professor Dickson gives a direct elementary proof of Jordan's canonical form of any linear homogeneous transformation (Traité des substitutions, pages 114–126, for the case of integral coefficients reduced modulo p, a prime), as extended to the general case of coefficients in an arbitrary field F (*American Journal*, 1902, pages 101–108). Let the characteristic equation have the roots $\alpha, \beta, \cdots$, with the multiplicities $a, b, \cdots$. The plan of the proof is to introduce a variables depending on α, b variables depending on $\beta, \cdots$, and to show that the new variables are independent. As the various sets are here introduced independently and not in succession, one avoids the necessity of the numerous normalizations made by Jordan (pages 120–123) and the intricate proof (page 121) that each set involves a single irrationality. The essential value of the canonical form lies less in the form of the explicit formulas than in the exact relation of the new variables to the irrationalities introduced. The paper will be offered for publication in the *Annals of Mathematics*.

18. In this note Professor Kasner separates the geometric from the analytic elements involved in Lie's well-known interpretation, and obtains a concrete representation by introducing a surface whose contour lines are the curves defined by the

differential equation. The slope of this surface visualizes the integrating factor.

19. In a former paper Professor Kasner showed that when a surface is represented conformally on a plane its geodesics are pictured by a family of curves whose central loci are straight lines. In the case of curves of given geodesic curvature it is shown that the corresponding loci are conics. When the curvature is varied the conics corresponding to a given point have a common focus and a common directrix.

20. With reference to a remark made by Professor Study in the April, 1907, number of the *American Journal of Mathematics* (page 101, note) Mr. Schweitzer observed that he has already shown that right-handedness is a descriptive relation which, with the aid of the indefinable point and ordered dyad, generates a descriptive geometry extensible, on the basis of the axioms, to the usual spatial projective geometry by well known methods.*

Mr. Schweitzer then briefly indicated some of the more important results of his researches in descriptive, projective and metrical (euclidean) geometry, which will be offered to the *American Journal of Mathematics* for publication in detail:

(1) There exists not only a fundamental linear relation (viz., the relation of Vailati) but a fundamental n-dimensional descriptive relation R_n ($n = 1, 2, 3, \cdots$). For $n = 3$ the relation is that of absolute right-handedness. Further, the relation R_n is n-dimensionally transitive, (*i. e.*, $xR_n a_2 \cdots a_{n+1}$, $a_1 R_n x a_3 \cdots a_{n+1}$, $\cdots a_1 R_n a_2 a_3 \cdots x$, imply $a_1 R_n a_2 \cdots a_{n+1}$) and alternating.

(2) Descriptive or projective order in n-dimensions ($n = 1, 2, 3, \cdots$) may be generated p-dimensionally ($p = 1, 2, 3, \cdots n$), so that the geometric spaces may be said to have the double index (p, n), $1 \leqq p \leqq n$. The preceding results have an important critical bearing on the discussion of geometric order given by Russel in his Principles of Mathematics.

(3) The descriptive n-dimensional systems generated by the relation R_n may be transformed into equivalent systems generated by the relation K_n,† the latter for $n = 3$ express-

* The relation of right-handedness referred to is the relation R_3 of the author, cf. BULLETIN, Nov., 1906, p. 79. The results indicated above were also demonstrated by the author in a paper read at the University of Chicago, Feb., 1905 ; at that time an undefined *class*, corresponding to the relation R_3, was used.

† Cf. BULLETIN, June and Nov., 1906.

ing concretely sameness of sense, or relative right-handedness. For $n \geqq 3$ Mr. Schweitzer showed that the corresponding n-dimensional K_n systems may be made "complete" for euclidean geometry by adding a certain number of axioms, the system of real numbers being introduced in a very simple manner. These "complete" systems constitute an elegant basis for Grassmann's Ausdehnungslehre and the exposition of the latter by Peano. In this connection it is of interest to consider a K_0 system of dimensionality zero, in addition to the K_n systems mentioned above.

21. In an earlier paper (read December 28, 1906), Dr. Griffin showed that Newton's law of attraction is the only one which permits the ellipse as a central orbit, and which satisfies the conditions: (1) The force is a single-valued function of the distance; (2) real motion is possible in all parts of the plane; and (3) there is no circle about the center of force, throughout which the force continually decreases toward the center. It has since been found that condition (1) is superfluous, as any law not satisfying (1) fails also to satisfy (2). In the present paper proof of this redundancy is given; and as a corollary thereto is obtained the theorem of Darboux and Halphen. *

22. It has been generally overlooked that the law of force, under which a given curve is described as a central orbit with a given constant of areas, cannot be uniquely determined from a knowledge of the position of the center of force. Dr. Griffin, treating this question in his second paper, considers especially a few of the infinitude of laws, other than that of Newton, for which a possible orbit is the ellipse with the center of force at a focus. Various properties of other trajectories than the original ellipse are pointed out.

23. This paper is the development of a communication made by Professor Davis to the Chicago section at its March meeting (see BULLETIN, June, 1907, page 436). In it are presented in order the notation with its adaptation to homogeneous coordinates and the effect of real and imaginary shifts of the axes; scalar and vector operations; the linear relation — central, non-central, and real; the quadratic relation as exhibited by the real and the imaginary circle and by other conics; the real cubic and its nine inflexions.

* *Comptes rendus*, vol. 84, pp. 760–762, and 936–941. Cf. also Tisserand, Mécanique céleste, vol. 1, pp. 36–42.

24. In this second paper, Professor Davis applies his method to three-dimensional space. He considers the planar relation, axial, non-axial but spatial, non-spatial; the linear relation, spatial and axial, spatial but not non-axial; combination of linear and planar relations; the cylinder; the sphere; the connection with Von Staudt's representation of imaginaries by involutions.

25. Dr. Sisam showed that the quartic surface $\phi(A, B, C, D) = 0$, ϕ, A, B, C and D being quaternary quadratic forms, which was considered by Durrande * in the *Nouvelles Annales* as the most general quartic surface, is not the most general surface of fourth order, but that a single condition must be satisfied by the coefficients. He also showed that the equation of the most general quartic surface can be written in the form $\theta(A, B, C, D, E) = 0$.

26. In Professor Snyder's second paper it is shown that if a plane curve of order n has not more than $2(n - 4)$ double points, the simplest curve to which it can be birationally reduced can be obtained by a Cremona transformation. The maximum number of basis points of a net of adjoint curves of order n can be found from any point P and its images under the transformation $x' = x$, $y' = \theta y$, $z' = \theta^n z$, $(\theta^{n^2-n+1} = 1)$. These theorems are applied to space curves, and it appears that the plane curves which are the projections of complete intersections of two surfaces of orders m, m' cannot be birationally transformed to curves of order less than $mm' - 1$.

28. In Professor Veblen's note are defined and studied what may be called continuous increasing functions of (finite and transfinite) ordinal numbers. They are such that x and $f(x)$ are both ordinals; if $x_1 < x_2$, then $f(x_1) < f(x_2)$; if $\{x'\}$ is the set of all ordinals less than an ordinal x of the second kind, then $f(x)$ is the least ordinal greater than every $f(x')$. A well-ordered set of these functions is constructed of which Cantor's exponential function ω^x is the first. The study of this set of functions leads to a classification of Cantor's ϵ-numbers.

29. Lüroth's quartic is characterized by containing the ten intersections of five secant lines; in fact it has an infinite system

* *Nouvelles Annales*, series 2, vol. 9, p. 410.

of inscribed quinquilaterals, whose lines are tangent to a conic. It is associated to a determinate Clebschian quartic which has every such quinquilateral for a polar polygon. Each of these quartics is characterized projectively by a single invariant condition; that of the Clebschian is a known simple invariant, but for the Lürothian the condition has not been worked out. Miss Miller and Professor White show that it can be expressed by equating to zero the resultant of six cubics homogeneous in six unknowns.

This form of quartic is discussed by Lüroth in volumes 1 and 13 of the *Mathematische Annalen;* at the end of his second article he speaks particularly of the present problem. Scorza, in volume 2 (third series) of the *Annali di Matematica,* and in volume 52 of the *Mathematische Annalen*, makes an exhaustive study of the relation between the Lürothian and the Clebschian by the use of the 36 even thetas of three arguments. A short paragraph in Wieleitner's Theorie der ebenen algebraischen Kurven höherer Ordnung treats Lüroth's curve geometrically.

30. In the *Transactions* for January, 1906, and January, 1907, Professor Fite discussed the manner in which the degree of an irreducible linear homogeneous group is determined by the abstract properties of the group. The discussion was limited for the most part to groups whose orders are powers of a prime. In the present paper the same question is considered for a somewhat different category of groups — namely, those in which every non-invariant commutator gives an invariant commutator besides identity. It is shown that the degree of such an irreducible group is uniquely fixed and an expression for the degree is given.

Besides the articles by the author already cited, reference may be made to Frobenius, "Ueber die Primfactoren der Gruppendeterminante," *Berliner Sitzungsberichte*, 1896, II, page 1343, and Burnside, "On the reduction of a group of homogeneous linear substitutions of finite order," *Acta Mathematica*, volume 28, page 369.

31. Given a linear homogeneous substitution of degree n and period m, whose coefficients are rational numbers, and such that there is no similar substitution of degree less than n, then n is a certain definite number-theoretic function of m. This function enables Dr. Ranum to determine all the finite periods of

n-ary linear substitutions having rational coefficients. A linear substitution is said to be reducible in a given field if it can be transformed into a reduced form, whose coefficients belong to the field. In the field of rational numbers it is found that every irreducible linear substitution of period m is of degree $\phi(m)$. By means of this and other considerations, all n-ary linear substitutions of period m with rational coefficients are easily classified as to their reducibility; finally a method is given of finding a rational canonical form of every such linear substitution.

32. Professor Stephens showed that certain curves, whose equations are of the form

$$t^n - xt^{n-1} + \mu t^{n-2} - \alpha\mu t^2 + \alpha\mu t - \alpha = 0$$

are easily constructed mechanically — the instrument used being similar in principle to that for the pentadeltoid. These curves are of class n and order $2(n-1)$. Each curve is circumscribed to an ellipse, touching it in n points. In any direction there are $n-2$ parallel tangents, the centroid of whose points of tangency is constant. The line at infinity is a double tangent. The cusps are n in number, real or imaginary. The special case $\mu = (n-1)/(n-3)$ exhibits some interesting properties.

33. The possible groups to which algebraic configurations of genus 4 belong have been determined by Wiman,* by means of collineations in space of three dimensions. Miss Van Benschoten completes Wiman's results by interpreting geometrically the various transformations when the configuration is expressed as a plane binodal quintic curve, and by deriving their equations. Every curve c_5^4 of order 5 and genus 4 is the plane projection of a space sextic R_6 lying on a quadric F_2 and a system of cubic surfaces which in particular cases may become cones K_3. An illustration is furnished by the octahedron group; it leaves a plane and its pole as to F_2 invariant. The six points of R_6 in this plane are points of tangency of lines through the pole. R_3 lies on 6 K_3, whose vertices form a complete quadrilateral in the invariant plane. The points of R_6 lie on the sides of the diagonal triangle; they are also in six-fold involution on

† Wiman, "Ueber die algebraischen Kurven von den Geschlechtern 4, 5, 6, " Stockholm Academy, *Bihang till Handlingar*, vol. 21 (1895).

the conic containing them. This configuration completely determines the axial and central involutions which leave the curve invariant, and indirectly defines the group. After deriving this and similar results for the other groups, the R_6 is projected into c_5^4 and the corresponding forms discussed.

34. In the problem of minimizing the integral

$$I = 2\pi \int y \sqrt{x'^2 + y'^2}\, dt$$

between the two fixed points A_0 and A_1, Miss Sinclair proves that either the catenary or the Goldschmidt discontinuous solution furnishes an absolute minimum with respect to rectifiable curves in the region $y \geqq 0$. She then discusses a discontinuous solution consisting of three curves which meet in a point A_2, a catenary through A_0 and A_2, a second catenary through A_2 and A_1, and a normal to the x-axis. She finds that with respect to similar systems of rectifiable curves in the region $y \geqq 0$ the above extremal system furnishes a relative minimum for the integral under the following necessary and sufficient conditions:

1) The vertices of the two catenaries are equidistant from the x-axis, and meet each other and the vertical line through A_2 at an angle of 120°.

2) $A_0(t = t_0)$ is limited in position by the inequality $t_0 > t_0^*$, where t_0^* is the unique negative solution of the equation $\coth t - t = 2 + \log \sqrt{3}$, and $A_1(t = t_1')$ is limited by the inequality $t_1 < t_0'$, where t_0' is the unique positive solution of the equation $\coth t - t = \coth t_0 - t_0 - \log 3$, and $A_0'(t = t_0')$ is the conjugate of A_0, and is the point in which the system touches the envelope of the set of extremal systems through A_0 which satisfy condition 1). A_0' may also be located by a Lindelöf construction. The value of I furnished by this discontinuous solution is for a certain region less than that given by the Goldschmidt solution, but never less than that given by the single catenary through A_0 and A_1. The system may be constructed by means of liquid films. Actual measurements of the limit of stability of the film correspond nicely to the theoretical values for the conjugate point.

35. The first aim of Dr. Fréchet's paper is to generalize the known results concerning linear functional operations (and particularly Hadamard's theorem) to cover the case where these

operations are defined not only, as before, in the field of continuous functions (of a single variable) but also in the far greater field of the functions which have an integral according to Lebesgue's definition. The second point is to prove that any of the operations defined in such a field can be determined by one continuous function $u(x)$ of a single variable, called characteristic function of the operation. The operation U_f is numerically known by means of the limit of one double integral when $f(x)$ and $u(x)$ are known. Conversely $u(x)$ is obtained by applying the operation to the function $\phi_y(x)$, $\phi_y(x)$ being equal to 1 when $0 \leqq x \leqq y$ and to 0 when $y < x \leqq 2\pi$.

36. The original object of Professor Greenhill's memoir was to provide the simplest method of calculating numerically the electromagnetic constant of a helix, in the ampère balance, designed by the late Viriamu Jones and Professor Ayrton for weighing the electromagnetic attraction and so arriving at an independent measure of the electrical units.

Incidentally it was requisite to coordinate the conflicting notation of various writers on the subject, by adopting Maxwell's Electricity and Magnetism as a standard, and to develop Landen's quadric transformation in order to reconcile results apparently discordant, so that the fundamental geometric meaning of Landen's transformation appears in the course of the electromagnetic theory.

The analytic complexity in the reduction of the elliptic integral in electromagnetism, as well as in most dynamical problems arises in consequence of the appropriate integral of the third kind being of the circular form in Legendre's terminology; the elliptic parameter of Jacobi is then a fraction of the imaginary period, so that the expression by the theta function implies a complex argument and a table of the theta function would not be of complete utility unless made in a triple entry form.

But in the practical problems of electromagnetism it is the complete third elliptic integral which is sufficient for a solution, and this, as Legendre has shown, can be expressed by integrals of the first and second kind, complete and incomplete, for which Legendre's Table IX provides the numerical value.

In working with the third elliptic integral it is found a practical convenience to retain it in the algebraical form, and to delay the adoption of a Legendrian, Jacobian, or Weierstrassian notation; the circular form of the complete integral will then fall into one of four classes, according to the sequence of the quantities involved.

The geometric interpretation of a formula has been explained wherever possible, when used for the expression of an electromagnetic quantity, as mutual induction, magnetic potential, vector potential, and so forth; and the advantage of the Stokes function in analytic simplicity has been emphasized over the ordinary potential function.

F. N. COLE,
Secretary.

ON A SPECIAL ALGEBRAIC CURVE HAVING A NET OF MINIMUM ADJOINT CURVES.

BY PROFESSOR VIRGIL SNYDER.

(Read before the American Mathematical Society, September 5, 1907.)

IN researches on plane curves a very fruitful configuration has been employed by Küpper* in obtaining particular curves having double points in abnormal position. His method consists in using as basis points part of the intersections of two curves which satisfy certain prescribed conditions; pencils and nets are then constructed having these basis points as nodes. The question naturally arises whether the same procedure can be employed in other cases, making use of all the constants in the system. In the following note it will be shown that in such cases the special series which are obtained cannot be employed to reduce the order of the curve. Incidentally, an illustration is furnished of configurations having a $g^2_{2(n-1)}$ (a linear series with two degrees of freedom and of order $2(n-1)$), although the curve cannot be reduced to order $2(n-1)$.

1. Through $n-1$ points on a straight line pass two general curves of order n, c_n, c'_n. These curves intersect in n^2-n+1 residual points through which can be passed ∞^2 curves of order n. Any two points P, Q of the plane will determine two pencils $c_n+\lambda c'_n=0$, $\phi_n+\lambda\phi'_n=0$ contained in this net, which can so be made projective that the locus of the variable intersections is the curve c_{2n} of order $2n$

$$(1) \qquad c_n\phi'_n - c'_n\phi_n = 0.$$

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*C. Küpper: "Ueber das Vorkommen von linearen Schaaren . . . ," *Sitzungsberichte der Böhmischen Gesellschaft*, Prag, 1892, pp. 264–272.