HERMITIAN FORMS WITH ZERO DETERMINANT.

BY PROFESSOR J. I. HUTCHINSON.

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VERY little has yet been attempted in the arithmetic determination of infinite discontinuous groups on two or more variables. The earliest memoir on this subject, by Picard,* treats of groups which leave certain ternary Hermitian forms invariant. Such forms can always be reduced, as Picard has remarked in a later paper,† to the form

$$\alpha u\bar{u} + \beta v\bar{v} - \gamma w\bar{w}, \qquad (\alpha, \beta, \gamma > 0), \tag{1}$$

if the determinant of the form is negative. The locus of singular (or limit) points for the group is the spread obtained by equating (1) to zero. It is the object of the present note to determine the kind of groups which have their limit points on such a quadric spread in case the corresponding Hermitian form has a zero determinant.

Let the given form f be represented by

$$f \equiv \sum_{i,\,k=1}^{n} f_{ik} x_i \bar{x}_k, \tag{2}$$

in which $\bar{x}$ is the conjugate of x, and the coefficient f_{ik} is the conjugate of f_{ki}. The determinant $D = |f_{ik}|$ of this form we assume to be zero and of rank k. Accordingly the elements of a certain column in D (the gth, say) are a linear combination of the corresponding elements of the other columns, that is,

$$f_{ig} = \sum_{\lambda} \bar{a}_\lambda f_{i\lambda} \qquad \left(\begin{smallmatrix}\lambda=1,\,2,\,\ldots,\,g-1,\,g+1,\,\ldots,\,n\\ i=1,\,2,\,\ldots,\,n\end{smallmatrix}\right),$$

in which the $\bar{a}_\lambda$ are finite and g is a suitably chosen integer.

If now we make the substitution

$$\begin{aligned} \eta_\lambda &= x_\lambda + a_\lambda x_g \quad (\lambda = 1, 2, \cdots, g-1, g+1, \cdots, n), \\ \eta_g &= x_g, \end{aligned} \tag{3}$$

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* "Sur une classe de groupes discontinues de substitutions linéaires, etc.," *Acta Mathematica*, vol. 1 (1881), p. 297.

† "Sur les formes quadratiques ternaires indéfinies à indéterminées conjuguées, etc.," *Acta Mathematica*, vol. 5 (1885), p. 121.

the form f reduces to

$$(4) \qquad \sum_{\lambda,\,\mu} f_{\lambda\mu}\eta_\lambda\bar{\eta}_\mu \quad (\lambda,\ \mu = 1,\ 2,\ \cdots,\ g-1,\ g+1,\ \cdots,\ n),$$

as may readily be verified by substituting in (4) the preceding expressions for η_λ and using the relations

$$\sum_\lambda f_{\lambda\mu}a_\lambda = \sum_\lambda \bar{f}_{\mu\lambda}a_\lambda = \bar{f}_{\mu g},$$

$$\sum_{\lambda\mu} f_{\lambda\mu}a_\lambda\bar{a}_\mu = \sum_\lambda a_\lambda f_{\lambda g} = \bar{f}_{gg} = f_{gg}.$$

The determinant of (4) is the first minor F_{gg} corresponding to f_{gg} in D. If the rank k of D is less than $n-1$, F_{gg} is zero and the process is continued until we reach a form whose determinant $M_{\mu\mu}$ does not vanish. This determinant is evidently a diagonal minor of D of order k, and accordingly its elements are common to the rows and columns nnmbered $\mu_1, \mu_2, \cdots, \mu_k$, these k integers being selected from the set $1, 2, \cdots, n$.

It is clear that by choosing different values for g in the transformation (3), and in the subsequent ones, the reduction could be made in a variety of ways and the determinant of the final form may be any non-vanishing diagonal minor of D of order k. We have now to show that, *in whatever way the given Hermitian form is reduced, the determinant of the final form always has the same sign.* This follows from a well-known theorem in determinants. For, let $M_{\nu\nu}$ be the diagonal minor of order k whose elements are taken from the rows and columns numbered $\nu_1, \nu_2, \cdots, \nu_k$. Denote by $M_{\mu\nu}$ the minor

$$\begin{vmatrix} a_{\mu_1\nu_1} & a_{\mu_1\nu_2} & \cdots & a_{\mu_1\nu_k} \\ a_{\mu_2\nu_1} & a_{\mu_2\nu_2} & \cdots & a_{\mu_2\nu_k} \\ \cdot & \cdot & \cdot & \cdot \\ a_{\mu_k\nu_1} & \cdots & & a_{\mu_k\nu_k} \end{vmatrix}$$

and by $M_{\nu\mu}$ the minor obtained from this by interchanging μ_i and ν_i $(i = 1, 2, \cdots, k)$. By interchanging rows and colums in $M_{\nu\mu}$, it is readily seen to be the numerical equivalent of $\bar{M}_{\mu\nu}$.

But since the determinant of the four minors thus obtained is zero,* we have

$$(5) \qquad M_{\mu\mu} M_{\nu\nu} = M_{\mu\nu} \overline{M}_{\mu\nu}.$$

The right member is positive or zero and hence if $M_{\mu\mu}$ and $M_{\nu\nu}$ be not zero they must have the same sign.

Suppose now that (2) has been reduced to the form

$$(6) \qquad \sum_{i, j=n-k+1}^{n} f_{ij} y_i \bar{y}_j,$$

the determinant of which $|f_{ij}|$ is not zero. A transformation on the x's which leaves the form (1) unaltered will be replaced by a corresponding transformation on the y's of the form

$$(7) \qquad \begin{aligned} y'_\lambda &= \sum_{\mu=\lambda}^{n} a_{\lambda\mu} y_\mu \qquad (\lambda = 1, 2, \cdots, n-k), \\ y'_i &= \sum_{j=n-k+1}^{n} a_{ij} y_j \quad (i = n-k+1, \cdots, n). \end{aligned}$$

A closer characterization of the group can be made sufficiently clear by an examination of the simplest case, $n = 3$. The general substitution of the group may be represented by

$$(8) \qquad x' = ax + by + cz, \quad y' = \alpha y + \beta z, \quad z' = \gamma y + \delta z.$$

Consider first the substitutions of the form

$$(9) \qquad x' = a'x + b'y + c'z, \quad y' = y, \; z' = z,$$

the totality of which form a subgroup of (8). As a group of transformations on the single variable x, it must be contained in the cyclic or the parabolic rotation groups.† It follows then that either b', c' are zero, or that a can have only the values ± 1 and the remaining terms in x' are expressible in the form

$$b'y + c'z = m\omega + n\omega'$$

* E. Pascal, Die Determinanten, Leipzig, 1900, p. 195. The determinant D might be called skew conjugate, and includes the symmetric and the skew-symmetric (after multiplying all the elements by i) as special cases. The theorem formulated in (5) would specialize in these cases as follows:

If a symmetric, or a skew-symmetric, determinant of order n is of rank $k < n$, then all diagonal minors of order k which do not vanish have the same sign.

† Fricke-Klein, Automorphe Functionen, I, p. 214 ff.

in which ω, ω' are arbitrary, fixed parameters and m, n are integers. Hence, *on replacing y, z by suitable variables ω, ω', the group* (8) *reduces to one of the forms*

$$(10) \qquad \begin{vmatrix} a & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \delta & \gamma \end{vmatrix}, \qquad \begin{vmatrix} \pm 1 & m & n \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{vmatrix}.$$

If the group is of the first type, it is completely reducible * and of no interest as a ternary group. If of the second type, the coefficients α, β, γ, δ must be integers such that $\alpha\delta - \beta\gamma = 1$, otherwise the combination of two substitutions would give one whose determinant does not have integers in its first row, and the group would contain infinitesimal transformations.†

The method of reasoning which we have applied to ternary groups is readily extended to the case of n variables and gives the result:

Either the group is completely reducible and has the form ‡

$$\begin{vmatrix} G & 0 \\ 0 & G' \end{vmatrix}$$

in which G' is the group of the reduced form (*6*) *and G is any discontinuous group in the remaining $n - k$ variables, or it has the form*

$$\begin{vmatrix} G_{11} & 0 & 0 \\ 0 & G_{22} & \mathrm{G}_{23} \\ 0 & 0 & G_{33} \end{vmatrix},$$

where G_{11} is any discontinuous group on p of the variables ($p < n - 2$), *and the other matrices have the form*

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*See W. Burnside, *Acta Mathematica*, vol. 28 (1904), p. 369, and A. Loewy, *Transactions*, vol. 6 (1905), p. 505.

† The group thus determined is of particular interest on account of its connection with the theory of elliptic functions. In fact, the Weierstrass functions constitute a complete system of invariants for the group. See Klein-Fricke, Modulfunctionen, vol. 2, p. 3 ff.

‡ For this very convenient symbolic representation see A. Loewy, *Transactions*, vol. 4 (1903), p. 44.

$$G_{22} = \begin{vmatrix} \pm 1 & 0 & 0 & \cdots \\ 0 & \pm 1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ 0 & \cdot \quad \cdot & \cdot & \pm 1 \end{vmatrix}, \quad G_{23} = \begin{vmatrix} 0 & 0 & \cdots & m_1 & n_1 \\ 0 & 0 & \cdots & m_2 & n_2 \\ & \cdot & \cdot & \cdot \quad \cdot & \cdot \\ 0 & 0 & \cdots & m_r & n_r \end{vmatrix},$$

$$G_{33} = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix},$$

in which m_i n_i a, β, γ, δ are integers and $\alpha\delta - \beta\gamma = 1$.

CORNELL UNIVERSITY,
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TWO TETRAEDRON THEOREMS.

BY PROFESSOR HENRY S. WHITE.

THE sphere and the tetraedron yield two combinations familiar to students of geometry, those in which one object is inscribed in the other; and one less well known, that in which the edges of the tetraedron are tangents to the sphere. A novel theorem upon the circumscribed tetraedron was propounded in 1897 by Bang and proved by Gehrke, and has been made the starting-point for extended developments by Franz Meyer * (1903) and Neuberg (1907).† It is this: If the contact point in each face of a tetraedron circumscribed about a sphere be joined by a straight line to each vertex in its face, then three angles at each contact point are equal respectively to the three formed at any other contact point. Or it may be stated thus: Opposite edges of a circumscribed tetraedron subtend equal angles at the points of contact of the faces which contain them.

While elementary proofs of this are interesting, a more elaborate deduction is of value here as suggesting a second theorem. It can be made to depend upon the well-known theorem from the projective geometry of a straight line, namely,

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* W. Franz Meyer: "Ueber Verallgemeinerungen von Sätzen über die Kugel und das ein- resp. umbeschriebene Tetraeder." *Jahresbericht der deutschen Mathematiker-Vereinigung*, 1903, p. 137.

† J. Neuberg: "Ueber die Berührungskugeln eines Tetraeders," *Jahresbericht der deutschen Mathematiker-Vereinigung*, 1907, p. 345.