That is, for any value of α the values of β terminate on a straight line perpendicular to $\bar{\delta}$. Likewise for any value of β the values of α terminate on the line

$$\alpha = -\frac{\beta\delta}{\bar{\gamma}} + t\frac{i}{\bar{\gamma}},$$

which is perpendicular to γ. This is equivalent to saying that if β terminates on the line through ξ at right angles to $\bar{\delta}$, α will terminate on the line through $-\xi\delta/\bar{\gamma}$ perpendicular to γ. Or in brief, to the line through θ/δ perpendicular to $\bar{\delta}$ corresponds the line through $-\theta/\bar{\gamma}$ perpendicular to γ, in the sense that any quaternion q whose projections terminate on these two lines in the A and B planes respectively, is *perpendicular* to the quaternion $r = \gamma + \delta j$.

6. The applications of this method to four-dimensional space are obvious, the A and B planes being two planes having only one point in common, the origin. The representation is in fact an adaptation of the descriptive geometry of four-dimensional space to the representation of quaternions as four-dimensional vectors, although this interpretation is not essential to the representation.

LOGIC AND THE CONTINUUM.

BY PROFESSOR EDWIN BIDWELL WILSON.

THE problem whether every set and in particular the continuum can be well ordered has attracted considerable attention since the days when G. Cantor first stated it in 1883.* In 1904 Zermelo offered an affirmative solution of the problem,† but his solution has not been generally regarded with favor and the discussion of the whole problem has now turned largely to a discussion of his solution. In a recent article he has summarized and discussed this discussion so fully that no repetition is called for at this time.‡ In entering so vast an arena of conflict, I would make no pretense of settling the dif-

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* *Mathematische Annalen*, vol. 21, p. 550.

† Zermelo, "Beweis, dass jede Menge wohlgeordnet werden kann," *Mathematische Annalen*, vol. 59, pp. 514–516.

‡ Zermelo, "Neuer Beweis für die Möglichkeit einer Wohlordnung," *ibid.*, vol. 65 (1907), pp. 107–128.

ficulties or of solving the problem; I merely desire to call attention to some general matters of logic which have an application to this problem and to others, and which appear to me to have been somewhat neglected. The introduction of a different point of view may be of service. My comments will at first be directed upon the idea of the categoricity or sufficiency of a set of postulates.

Although the method of nominal definition from logical elements is regarded in many quarters as preferable to the determination by means of postulates, when a specified field of mathematics is to be investigated, there still remains so much difficulty in the way of the purely nominal definition of mathematics that the determination by means of postulates cannot as yet be considered as obsolete, perhaps not even as obsolescent; and this method apparently must obtain in laying the foundations of logic itself, whether or not the nominal definition is finally adopted for the subsidiary sciences. The ideas that a set of postulates should not be redundant and must not be contradictory have been widely exploited. There is, however, another idea which so far as I am aware was first clearly stated and emphasized by Huntington and which has not yet received a great measure of recognition. This idea is that a set of postulates may be sufficient or categorical.*

A set of postulates $[P] = P_1, P_2, \cdots, P_n$ connecting a set of undefined symbols $[S] = S_1, S_2, \cdots, S_m$ is said to be categorical if between the elements of any two assemblages, each of which independently contains the undefined symbols $[S]$ and satisfies the postulates $[P]$, it is possible to set up a one to one correspondence which preserves the significance of the undefined symbols. This last restriction is vital — it ensures a sort of logical conformality in the correspondence. For it is ob-

*Huntington, "A complete set of postulates for the theory of absolute continuous magnitude, *Transactions of the American Mathematical Society*, vol. 3 (1902), pp. 264–279. "Complete sets of postulates for the theories of positive integral and positive rational numbers," *ibid.*, pp. 280–284. "Complete sets of postulates for the theory of real quantity," *ibid.*, vol. 4 (1903), pp. 358–370. In these earlier articles Huntington uses the term sufficient; in later communications, "A set of postulates for real algebra, comprising postulates for a one-dimensional continuum and for the theory of groups," *ibid.*, vol. 6 (1905), pp. 17–41, and "The fundamental laws of addition and multiplication in elementary algebra," *Annals of Mathematics*, ser. 2, vol. 8 (1906), pp. 1–44, he adopts somewhat hesitatingly the term categorical from a paper of Veblen's and points out, *Annals*, p. 26, that Peano had and apparently realized that he had a categorical set of postulates for positive integers as early as 1891.

vious that if any set of elements $[K]$ which satisfies the postulates $[P]$ on the symbols $[S]$ may be put into a one to one relation with a set of elements $[K']$, it must be possible to interpret $[P]$ and $[S]$ in the set $[K']$ by a mere transference from the original set $[K]$. This procedure is of frequent occurrence in mathematics,* but it has little to do with categoricity for the reason that $[K]$ and $[K']$ are not determined independently.

It is not always desirable and indeed not always possible to obtain a set of postulates which shall be categorical: for it may well happen that the systems to be determined are such that not even a one to one correspondence between their elements is available, to say nothing of the preservation of the interpretation of the symbols $[S]$. For instance in defining a group it would clearly be unwise so to restrict the definition that all groups were of equal multitude. And in defining a geometry in a space of constant curvature, there is little to be gained by a restriction which specifies the magnitude of the curvature unless it be zero. It is, however, advisable to have a categorical determination of euclidean geometry and of the continuum and of real algebra. This has been accomplished by Huntington and Veblen.† In case these domains of mathematics are defined nominally it is equally important that the definition shall be sufficiently detailed to ensure an equivalent precision in the system defined.

Huntington gives a subsidiary definition or explanation of the idea of categoricity wherein he asserts that if a set $[P]$ of postulates on the undefined symbols $[S]$ is categorical, then every proposition concerning $[S]$ must be deducible from the postulates $[P]$ or be in contradiction with them.‡ In interpreting this statement it should be noted that not every conceivable property of the defined set $[K]$ is asserted as either deducible from $[P]$ or in contradiction with $[P]$, but only such properties as are expressible solely in terms of $[S]$. Thus the statement that the rational numbers greater than 0 are red and those less than 0 are blue is neither in contradiction with

* A little reflection will show that even here the value of the transference of properties from $[K]$ to $[K']$ is almost nil except as the transferred properties may be related to properties which $[K']$ independently possesses.

† References to Huntington have been given. Veblen, "A system of axioms for geometry," *Transactions of the American Mathematical Society*, vol. 5 (1904), pp. 343–384.

‡ See his last paper cited, or Veblen's.

Huntington's set of postulates nor deducible from them, because the statement contains terms (red, blue) which are not defined or nominally definable in the system; moreover, any statement about "alternate rational numbers," where alternate refers to the numbers in their natural order, is neither compatible nor incompatible with the postulates, because that statement is not a proposition at all but merely a meaningless collocation of words.*

I have advisedly used the terms compatible with $[P]$ and incompatible with $[P]$ in the respective places of deducible from $[P]$ and in contradiction with $[P]$. Is this a justifiable change? It is readily seen that any proposition (statement which may be either true or false but not meaningless) phrased in terms of $[S]$ or derived symbols must be either compatible with $[P]$ or incompatible with $[P]$: for, owing to the categoricity of the determination, it is impossible that the proposition should be true of one set $[K]$ and false of another set $[K']$ both of which satisfy the postulates. Does it follow that every such proposition must either be deducible from $[P]$ or in contradiction with $[P]$? This question, this suggestion that compatibility and deducibility may not be the same when applied to categorically determined systems is vital in logic and requires careful discussion.†

In the first place it is evident that if the determination of a system is not categorical, then compatibility and deducibility cannot be entirely equivalent. This follows from the very idea of compatibility, to wit, that any proposition phrased in terms of the undefined symbols shall either be true for all the systems determined by $[P]$ or be false for them all. Any individual postulate of a set $[P]$ which is not redundant is neither compatible nor incompatible (or is both compatible and incompatible, if pre-

* The difficulties that are involved in meaningless statements have been recognized since very early times; for instance, the case of the liar who says he is a liar. Such is also in all probability the difficulty with the class of all classes; see Zermelo, "Untersuchungen über die Grundlagen der Mengenlehre," *Mathematische Annalen*, vol. 65 (1908), pp 261–281.

† Huntington quotes a remark of Davis on this subject in a later paper "A set of postulates for ordinary complex algebra," *Transactions*, vol. 6 (1905), p. 210, footnote †; Veblen dismisses it with the statement that any additional postulate would have to be considered redundant even were it not deducible from the categorical set by a finite number of syllogisms, *loc. cit.*, p. 346. As I do not know what he means by this, I am unable to say whether or not his position is in practical accord with that which I shall set forth in the following paragraphs.

ferred) with the other postulates of [P].* In the second place consider a system determined categorically. Here every proposition phrased in terms of [S] is either compatible or incompatible with [P]. What, however, does the word deducible mean? The meaning is entirely relative to the system of logic which is available for drawing conclusions from the set of primitive propositions [P]. Some may consider that the human mind has instinctively at its disposal all valid methods of deduction. This is a tremendous postulate, and one entirely devoid of other than sentimental value. In fact, if it leads to the abandoning of the research for valid methods of deduction, it is dangerous and worse than useless. It is an essential of the modern attitude in logic that the deducer should state distinctly his form of inference. Hence deducible cannot be regarded as equivalent to compatible.

It is clear that in an ideal perfection of logic compatibility and deducibility would be equivalent for categorically defined systems. That state of perfection appears at present to be very remote. The constant attempt to bring compatibility and deducibility into coincidence will undoubtedly do much to advance the condition of logic, just as in the days before Huntington so clearly stated the idea of categoricity that attempt did much to advance mathematics.† It appears to me, however, that it may be a distinct gain in precision and hence a considerable advantage to admit the following orienting propositions, namely:

So long as there is an unsolved problem of pure mathematics the solution may be lacking

1° because the class of objects to which the problem belongs is not sufficiently determined or

2° because the available logical methods of deduction are insufficient; but in case the class of objects is categorically determined, 2° alone applies.‡

To take some examples. Fermat's problem of the solution of

$$x^n + y^n = z^n, \quad n > 2,$$

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* This statement is capable of a small amount of generalization by including under proper restrictions theorems as well as postulates.

† For instance, the development of noneuclidean geometry could be interpreted as a somewhat unconscious striving in this direction.

‡ There is, of course, the possibility that the lack of solution is due to our failure to perceive the method by which the logical principles already in our possession should be combined to reach the desired result. Logically, however, this is a very different sort of failure from either of those mentioned, and it has been thought better not to refer to it in the text.

in terms of integers cannot fail of solution because of insufficient determination of the class of integers, inasmuch as integral algebra has been established upon a categorical basis. From the nature of this problem it appears likely that the lack of solution is not due to the incompleteness of the methods of deduction, but rather to a failure properly to combine the existing methods. Goldbach's theorem that any even number may be written as the sum of two primes cannot, for a similar reason, fail of proof owing to an insufficiency of determination. Peano states, * if I understand his inflexionless latin, that this theorem fails of a satisfactory demonstration because, like the theorem that any infinite set contains an enumerable subset, it depends on the application of the principle of Zermelo (see below). Thus for Peano and Zermelo deducible has different meanings; and any one is at liberty to side with either party to the dispute.† It may, however, be pointed out that, by some method of combining forms of deduction which both would admit, Goldbach's theorem might perhaps be demonstrated: but it is difficult to see how such a reconciliation could be operated in regard to the theorem on the existence of an enumerable subset in any infinite set.

In discussing the problem of the well-ordering of the continuum from the point of view I have here set forth, it is first necessary to ask if the definition of well-ordering can be phrased in terms of the undefined symbols introduced in the categorical determination of the continuum. On comparing Zermelo's definition of well-ordering and Huntington's set of postulates,‡ it appears that the definition can be so phrased without introducing new symbols other than the general logical constants introduced in framing the postulates. Hence Zermelo has no right to add any new postulate expressing a property of the continuum. As a matter of fact, the postulate that he does add concerns classes in general and the usage he makes of it and his own statements show that what he has done is to demand a new postulate or principle of logic. Thus to prove

* Peano, "Supra theorema de Cantor-Bernstein," *Revista de Matematica*, vol. 8 (1906), pp. 136–157; see especially p. 146.

† As Zermelo, in common with most other investigators, does not state what his logical postulates are, it is impossible to say just what deducible does mean for him. This difficulty, which is scarcely felt at all in ordinary reasoning, becomes very troublesome when one tries to follow the great number of different writers on transfinites.

‡ Zermelo, *loc. cit.*, *Annalen*, 65, p. 111. Huntington, *loc. cit.*, *Transactions*, 6, p. 31.

that the proposition that the continuum is well ordered is compatible with the postulates concerning the continuum, he enlarges his logic and shows that any set is well ordered.

In view of the facts that it may be doubted whether our logic is yet complete and that Zermelo's postulate is apparently not in contradiction with the other logical postulates, it is difficult to see how any one can deny him the right to proceed as he does. The only question appears to be whether his method is valuable. And certainly, if such an elementary theorem of arithmetic as Goldbach's theorem cannot be proved or disproved without some such addition as Zermelo makes to our logical procedure, it seems as if some such addition were imperative; for however one may hesitate about committing himself to the intricacies of the continuum, he surely would not indefinitely consign Goldbach's theorem to the class of non-deducibles. It therefore is proper to examine the new principle of logic and its consequences.

The first statement of Zermelo's principle is: For an infinite ensemble of sets it is possible to find a correspondence which correlates to each set a single element of that set.* For a finite ensemble of sets the proposition is readily admitted. The same is true of the later formulation of the postulate: A set S which is divided into subsets $A, B, C, \cdots$ each containing at least one element but containing no elements in common contains at least one subset S_1 which has just one element in common with each of the subsets $A, B, C, \cdots$.† To this postulate Peano objects that: one may not apply an infinite number of times an arbitrary law by virtue of which one correlates to a class some member of that class.‡ Here are two postulates by two different authorities; the postulates are contradictory, and each thinker is at liberty to adopt whichever appears to him the more convenient.

Lebesgue has recently pointed out the high degree of unattainability which characterizes the correspondence postulated by Zermelo. § I should like to call attention to another awkwardness of the theory. Consider at first a finite set of elements, say $M(1, 2, 3, 4)$, and follow out the statement of Zermelo's

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* Zermelo, *loc. cit.*, *Annalen*, 59, p. 516.

† Zermelo, *loc. cit.*, *Annalen*, 65, p. 110.

‡ Peano, *loc. cit.*, *Revista*, 8, p. 145. Peano had taken this position in an earlier communication in 1890, *Mathematische Annalen*, vol. 37, p. 210.

§ Lebesgue, "Contribution a l'étude des correspondances de M. Zermelo," *Bulletin de la Société Mathématique de France*, vol. 35, pp. 202–212.

fundamental theorem : Ist durch irgend ein Gesetz jeder nichtverschwindenden Untermenge einer Menge M eines ihrer Elemente als "ausgezeichnetes Element" zugeordnet, so besitzt die Menge $\mathfrak{U}(M)$ aller Untermengen von M eine und nur eine Untermenge $\mathbf{M}$ von der Beschaffenheit, dass jeder beliebigen Untermenge P von M immer ein und nur ein Element P_0 von $\mathbf{M}$ entspricht, welches P als Untermenge und ein Element von P als ausgezeichnetes Element enthält. Die Menge M wird durch $\mathbf{M}$ wohlgeordnet.* To avoid all the difficulties of following the author's deduction of this theorem, let it be assumed that the theorem is properly deduced.

In the first place it is necessary to construct the set $\mathfrak{U}(M)$ of all subsets of M. This is the set

1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234, 1234.

The number of elements in this set is $4 + 6 + 4 + 1 = 15$. More generally if M had contained n elements, the set $\mathfrak{U}(M)$ would have contained

$$n + \frac{n(n-1)}{2\,!} + \frac{n(n-1)(n-2)}{3\,!} + \cdots + n + 1 = 2^n - 1$$

elements. Next it is to be supposed that in each of these sets some element has been picked out as ausgezeichnet. This may be accomplished by printing that element of each set in red. The number of ways in which it may be accomplished is 20736. In case the number of elements had been n, the number of ways would have been

$$(a) \qquad 2^{\frac{n(n-1)}{2!}} \cdot 3^{\frac{n(n-1)(n-2)}{3!}} \cdots\cdots (n-1)^n \cdot n.$$

The next step is the assertion of the existence of the set $\mathbf{M}$ of the specified properties. This set will depend in large measure upon the particular way in which the ausgezeichnetes element of each set has been chosen. If it were agreed to print the smallest integer in each set in red, the set $\mathbf{M}$ would be

1234, 234, 34, 4,

and the corresponding way in which M would be well ordered would be 1, 2, 3, 4. However, by the method outlined by Zermelo, loc. cit., page 110, it is clear that with a proper dis-

* Zermelo, *loc. cit.*, *Annalen*, 65, p. 108.

tribution of the red ink it would be possible to obtain a set **M** which would give *M* any one of its possible 4 ! arrangements. In like manner if *M* contained *n* elements, a proper selection of the red elements would give rise to an **M** which would ensure to *M* any of its *n* ! possible orders.

As a matter of fact, it may be seen from Zermelo's reasoning that **M** must contain *M*, and at least one of the subsets of *M* in which only one element of *M* is missing, also at least one of the subsets of this subset where just one additional element is missing, and so on. Thus suppose that **M** were picked in accordance with this principle as

$$1234, \ 134, \ 13, \ 3.$$

The corresponding ordering of *M* would be 2, 4, 1, 3, and it would not be hard to assign a red element in each subset of *M* on this basis. This assignment could probably be made in an enormous variety of ways; for the number of ways in which red elements may be selected is (*a*), in this case 20736, whereas the possible arrangements for *M* number only *n* !, in this case 24.

Thus in the case of the well-ordering of a finite set a very complicated mechanism has been introduced which suffices not only to order the set, but to order it in all possible ways and moreover to accomplish this ordering in an even vastly greater number of ways. One is tempted to say that not only has the author proved his proposition (for finite sets), but so much more than proved it as to render the conception of well-ordering almost worthless. In view of the fact that there are many who do not believe in the transfinite cardinals at all and that those who do believe find it no easy matter to escape error in manipulating these numbers, one must be very cautious in extending to infinite sets a method of reasoning analogous to that here employed with reference to finite sets. Some general considerations may, however, be worth notice in a tentative way.

Suppose that *M* is the ensemble of elements in the continuum. Let their number be $\aleph_1$.* The set **U**(*M*) according to those who believe in such things would have a greater number, say $\aleph_2$. The very introduction of this set into the reasoning on the well-ordering of the continuum implies the use of something

* I do not wish to imply the existence of the transfinite cardinals: the three aleph symbols here introduced may be regarded as notations for hypothetically existing entities.

which is more transfinite than is necessarily called for by the theorem to be proved. And this greater transfinitude must be in working order if Zermelo's correspondence is to be obtainable. What might be the order of infinitude which shall suffice for specifying the number (a) of ways in which the correspondence may be set up is a question which will be left to the reader. There are, however, indications which point with considerable assurance to the inference that if Zermelo's reasoning were followed to its (logical?) conclusion, it would result in establishing the fact that the continuum may be well ordered in something like $\aleph_1!$ ways. If so, is the theorem worth having? Does it mean anything?

To get a glimpse of the bearing of these questions it may be well to look at the problem of well-ordering the positive rational numbers. The method of ordering them by telling them off against the positive integers is well known and is valuable. Suppose Zermelo's method were followed. Here $\mathbf{U}(M)$ would be that set which is introduced in defining the continuum — a set of greater infinitude than M. Suppose next that the correspondence required by the author were set up, the set $\mathbf{M}$ picked out, and the ordering accomplished. In view of the evidence that this could all be accomplished (if at all) in a high infinitude of ways, is it evident that any more (and perhaps even less) has been accomplished than would have been accomplished by the simple statement that to order and well-order the positive rational numbers it is merely necessary to pick out one of them and set it down and then pick out another and set that down next to the former, and so on?*

To sum up this discussion I should say:

1° that Zermelo is right in his contention that he may add his postulate to the existing logical system; that to deny him that privilege would be to put an embargo on the development of logic and to assume a completeness of our logical system which is quite unwarranted in view of past developments and future possibilities:

2° that the use he makes of his principle is to render a hard thing harder, a transfinite condition more transfinite; that there still remains so much of an arbitrary nature in the determination of the correspondence as to make his theorem on well-ordering

* This idea has been stated by Borel, "Quelques remarques sur les principes de la théorie des ensembles," *Mathematische Annalen*, vol. 60, pp. 194–195. For comment, see the next footnote.

of little more value than the statement that the elements of any set may be plucked out seriatim and set in a row; that he applies this latter principle (except for the seriatim plucking) to a case which really transcends the original set in infinitude:

3° That the well-ordering of any set is of practically no significance and is quite worthless apart from an algorithm which accomplishes the ordering — an algorithm which shall not require an operation which transcends the cardinal number of the given set.*

It should be added that, entirely apart from the problem of well-ordering, the application of Zermelo's principle to the construction of a general theory of sets (Mengenlehre) along the lines he is now following is an interesting study which may contribute in no small measure toward the solution of some of the fundamental questions of logic and may easily result in the general adoption of some form of the principle which he has introduced explicitly and which has tacitly been used to a very considerable extent with no overwhelmingly fallacious results. It seems far from certain that we shall never need a more general principle of reasoning than that contained in the present acceptation of the scope of mathematical induction.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS., *April*, 1908.

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* In forming any conclusions concerning Zermelo's problem one must not overlook the keen remarks made on both sides of the question by several authors and published under the title "Cinq lettres sur la théorie des ensembles" in the *Bulletin de la Société Mathématique de France*, vol. 33, pp. 261–273. My conclusions here stated agree very largely with those of Borel and Baire in those lettres; and hence there would be little need of my making the restatement that I do, were it not for the fact that I feel that I have treated the matter somewhat more in detail and in a way less open to objection on the ground of subjectivity or prejudice. Hadamard pointed out in his first letter (and this has an application to the last statement in 2°) that there was a difference between picking out one element after another from an infinite set and setting them in order on the one hand and selecting an element from each one of an infinity of independent sets on the other hand. If that be granted (it is not granted in the present case by all who have commented on the subject), I still fail to see how there is any great difference between plucking one element after another from the continuum and selecting an element from each one of the subsets of the continuum, especially in view of the fact that the set of subsets is of greater number (Mächtigkeit) than the continuum. It is the application of the principle of selection to this set of greater number which particularly impresses me; and in addition to this, the so great freedom of choice which is to be found in the selection. The freedom appears quite sufficient to enable the well-ordering to be likened to plucking one element after another. So far as I am aware, these two fundamental considerations have not been taken up by previous writers. This, with the discussion of logic which occupies the first half of the paper, may perhaps be a sufficient justification for my present communication.

Postscript. — Since the above was sent to the printer, Schoenflies's report on Die Entwickelung der Lehre von den Punktmannigfaltigkeiten, 2ter Teil, has come to hand and should be included among the references cited in this paper. On page 36, paragraph 3, the author states a conclusion which is practically that of Borel in the *Annalen*, loc. cit., volume 60, namely, that Zermelo's proof can be characterized as establishing the equivalence of the principle of selection and the principle of well-ordering. That is to say, if a given set is well-ordered a correspondence such as Zermelo postulates may be set up between the subsets and the elements of the set by simply associating to each subset its first element, and Zermelo's proof shows that conversely if the principle of selection is admitted the given set may be well-ordered.

I am by no means sure that these statements bring the two principles into equivalence, as Borel and Schoenflies assume. It should be remembered that if a set is well-ordered we can actually associate to *any* subset an ausgezeichnetes element, whereas the postulate of selection requires us by an arbitrary process to associate to *all* the subsets an element. As far as finite sets go, the distinction between any and all is not so great as not to be readily overlooked, although logicians regard a postulate concerning any element of a set as far simpler than one concerning all elements. When, however, it is a question of infinite sets and particularly of non-enumerable sets the difference between any and all is vast, and many statements or proofs, unexceptionable when phrased in terms of any, are by no means acceptable when all replaces any.

In view of these considerations and the oft-repeated fact that Zermelo's principle must be applied to a set of greater number than the one to be ordered, I should seriously doubt the equivalence between his principle and that of ordering and should feel quite justified in claiming that it was logically and mathematically preferable to take the principle of ordering as a postulate instead of the principle of selection in the general form in which Zermelo states it.

E. B. W.

May, 1908.