THE INVERSE OF MEUSNIER'S THEOREM.

BY PROFESSOR EDWARD KASNER.

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MEUSNIER'S theorem, relating to the curves drawn on an arbitrary surface

$$\text{(I)} \qquad f(x, y, z) = 0,$$

was extended by Lie * to the curves satisfying any Monge equation of the first order

$$\text{(II)} \qquad f(x, y, z, y', z') = 0,$$

where primes denote differentiation with respect to x. In this note we show that the theorem is valid in the more general case of curves defined by any equation of the form

$$\text{(III)} \qquad Ay'' + Bz'' + C = 0,$$

where A, B, C are arbitrary functions of x, y, z, y' z'; and that no further extension is possible.

Our problem is to find the most general system of space curves with the *Meusnier property*. This deals with the curvature of the curves of the system which pass through a common point O, in a common direction, and may be stated in any one of the following equivalent ways:

1. The radius of curvature varies as the sine of the angle between the osculating plane and a fixed plane.
2. The circles of curvature generate a sphere.
3. The locus of the centers of curvature is a circle through the point O.
4. The locus of the inverse centers of curvature is a straight line.

It will be convenient to use the last statement, sometimes referred to as Hachette's theorem. Taking the origin at the given point O, and the axis of X along the given direction, we find for the center of curvature

* *Leipziger Berichte*, vol. 50 (1898), p. 1; *Math. Annalen*, vol. 59 (1904), p. 299.

$$(4) \qquad X_1 = 0, \quad Y_1 = \frac{y''}{y''^2 + z''^2}, \quad Z_1 = \frac{z''}{y''^2 + z''^2}.$$

The inverse center of curvature, *i. e.*, the inverse of this point with respect to a unit sphere about O, is

$$(5) \qquad X = 0, \quad Y = y'', \quad Z = z''.$$

The Meusnier property requires that the point X, Y, Z, for the curves considered, shall be confined to a straight line. Let this line, which is necessarily in the plane normal to the given lineal element, be

$$(6) \qquad X = 0, \quad AY + BZ + C = 0.$$

Then, from (5), the relation between y'' and z'' must be of the form

$$(\text{III}) \qquad Ay'' + Bz'' + C = 0.$$

The line (6) is fixed for a given lineal element, but may vary when the element varies. Hence A, B, C are arbitrary functions of x, y, z, y', z'.

THEOREM. *The system of curves defined by any differential equation of the form (III) possesses the Meusnier property, i. e., the locus of the inverse centers of curvature of those curves of the system which have a lineal element in common is a straight line. Conversely, every system possessing this property satisfies an equation of the form (III).*

Equations of the types I and II give rise by differentiation to special cases of the type III. More generally we may start with any two-parameter family of surfaces

$$(7) \qquad f(x, y, z, a, b) = 0,$$

or any one-parameter family of Monge equations of the first order

$$(8) \qquad f(x, y, z, y', z', a) = 0,$$

and derive equations of the form III. The systems of curves defined by (7) and (8) constitute special species (the second including the first) of the general class III.

The Meusnier property, although stated in metric form, really belongs to the general geometry whose group is composed of all (analytic) point transformations. This follows from the fact that in any point transformation the new values of y'' and z'' are linear and integral in the original values. The type III is, therefore, invariant. The species (7) and (8) are also invariant under this group.

The case where the absolute term C vanishes, so that the equation III reduces to

$$(9) \qquad Ay'' + Bz'' = 0,$$

is metrically exceptional. The statement 1. of the Meusnier property then becomes illusory since all curves having an element in common here have a common osculating plane. The sphere of statement 2. reduces to this plane, and the loci of 3. and 4. reduce to the same straight line passing through O. The type (9) gives the most general system of space curves such that curves having an element in common also have a common osculating plane. This type is not invariant under the group of all point transformations but is invariant under the projective group.

In connection with the general type III we may develope a theory of *straightest curves* which includes as special cases the theory of geodesics on a surface and the Hertzian theory of path curves in the non-holonomic motion defined by a Pfaffian equation. A curve of the system represented by an equation III is here termed a straightest curve of the system when its curvature at any point is less than (or at most equal to) the curvature of the other curves of the system which pass through that point in the same direction.

The differential equations of these curves can easily be derived by writing down the conditions that the curvature, which is a known function of x, y, z, y', z', here regarded as fixed, and of y'', z'' regarded as variable, but subject to the relation III, shall be a minimum. We prefer, however, to derive the equations from the fact that the locus of the inverse centers of curvatures is a straight line. When the lineal element has its direction given by any values of y' and z', the associated straight line is found to be

$$(10) \qquad \begin{aligned} &X + y'Y + z'Z = 0, \\ &(Ay' + Bz')X - AY - BZ - \frac{C}{1 + y'^2 + z'^2} = 0. \end{aligned}$$

For a curve of least curvature, the osculating plane, whose general equation is

$$(11)\qquad \begin{vmatrix} X & Y & Z \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0,$$

must meet the line (10) in the point nearest to O. This means that (11) must be perpendicular to (10), so that

$$(12)\qquad \begin{vmatrix} Ay' + Bz' & -A & -B \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0.$$

Combining this with III, we find, as our final equations,

$$(13)\qquad \begin{aligned} y'' &= \frac{-C(A + Ay'^2 + By'z')}{A(A + Ay'^2 + By'z') + B(B + Bz'^2 + Ay'z')}, \\ z'' &= \frac{-C(B + Bz'^2 + Ay'z')}{A(A + Ay'^2 + By'z') + B(B + Bz'^2 + Ay'z')}. \end{aligned}$$

Our result may be stated as follows:

*In any system of curves of type (III) there exists a four-fold infinitude of straightest curves, i. e., curves of least curvature, one passing through each point in each direction. The differential equations of these curves are given by (13).**

The systems (13) which thus present themselves are of perfectly general character. For, given any system of ∞^4 space curves, one through each lineal element, its differential equations may be written

$$(14)\qquad y'' = g(x, y, z, y', z'), \quad z'' = h(x, y, z, y', z').$$

By comparing these with (13), the ratios $A : B : C$, and hence an equation of type III, may be found. The inverse problem which thus arises (determination of a system III from its straightest lines) is, in general, capable of only one solution.†

* In the case of type I there are ∞^2 such curves, namely the geodesics. In type II there are ∞^3 straightest curves.

† An exception arises when the assigned system (14) consists of the straight lines $y'' = 0$, $z'' = 0$. The system III may then be any one of the form (9).

It is of interest to notice that the type III presented itself first in connection with the general transformation theory * of the lineal elements of space. Such a transformation is defined by setting x_1, y_1, z_1 y_1', z_1' equal to arbitrary functions of x, y, z, y', z'. Lie showed that the only case in which every union is turned into a union is the extended point transformation. There may, however, be other transformations which convert some unions into unions. Examples of this sort have been given by Lie in which the unions considered are the curves of a minimal complex or some other Monge equation (II). The general result is as follows:

The most extensive systems of curves that can be converted into curves by element transformations which are not merely extended point transformations are those defined by differential equations of type III, i. e., precisely those characterized by the Meusnier property.

For any system of type III transformations of this sort may be found. If a system not satisfying an equation III is transformed into curves, the element transformation is necessarily an extended point transformation.

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ON THE DISTANCE FROM A POINT TO A SURFACE.†

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IN a recent number of the BULLETIN,‡ Professor Hedrick has called attention to the fact that the normal distance from a given point to a surface may be a minimum among the distances in every normal plane section without however being a minimum among the distances to the surface. In this connection it may be of interest to observe that for surfaces of a very general type the phenomenon in question can occur only when the given point is a principal center of curvature. In fact, if the equation

* Cf. abstract of the author's paper on this subject in the BULLETIN, vol. 10 (1904), p. 492.

† The theorems established in a previous note (BULLETIN, vol. 13, p. 447) are not new, as I then believed; they are to be found in section 60 of Goursat's Cours d'Analyse. I take this opportunity of acknowledging Professor Hedrick's kindness in calling my attention to the fact.

‡ BULLETIN, April, 1908, p. 321.