through two conical points the (2, 2) correspondence is defined by the pencils through the nodes. The second also refers to quartic surfaces, those having two nets of hyperelliptic curves. The (2, 2) correspondence is defined by the lines joining the points of the canonical g_2^1. The third concerns the systems of bitangents on any surface which is complete focal surface of two or more congruences.

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BÉZOUT'S THEORY OF RESULTANTS AND ITS INFLUENCE ON GEOMETRY.

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BY PRESIDENT HENRY S. WHITE.

THE accepted truths of today, even the commonplace truths of any science, were the doubtful or the novel theories of yesterday. Some indeed of prime importance were long esteemed of slight importance and almost forgotten. The first effect of reading in the history of science is a naive astonishment at the darkness of past centuries, but the ultimate effect is a fervent admiration for the progress achieved by former generations, for the triumphs of persistence and of genius. The easy credulity with which a young student supposes that of course every algebraic equation must have a root gives place finally to a delight in the slow conquest of the realm of imaginary numbers, and in the youthful genius of a Gauss who could demonstrate this once obscure fundamental proposition.

The first complete proof, by Gauss, that rational algebraic equations have roots either real or imaginary dates back only to 1799. That part of algebra that is concerned with equations is accordingly for the most part modern, recent indeed, as compared for instance with the plane geometry of lines and circles. Before Gauss, it is true, much had been done in actually solving equations of the lower orders and in the theory of symmetric functions of the roots. After him also the question of the arithmetical character of the roots required not only a Galois to penetrate its mystery, but also a Liouville and a Jordan to expound the marvellous theory that Galois had created.

In the general mathematical consciousness of the present day is dominant, even when not active, the knowledge revealed by Gauss and Galois; and this widespread understanding of the nature of the solutions of a single equation constitutes the goal toward which tended for three centuries the labors of Tartaglia, Cardan, Ferrari, and the long train of emulous scholars culminating in Lagrange and Abel. The theory of the single equation in one unknown is now, if not completed in all its details, yet at least finished in foundation and in the framework of its superstructure.

Equations in two variables have received no less attention, but from the nature of the case it must be long before their theory could reach any corresponding stage of completeness. Geometrically, a set of discrete points on a line has obviously fewer properties to investigate than a curve in a plane. The projective covariants of the one are all, like itself, sets of discrete points; while those of the other are of four or more different kinds: points, lines, curves, and line-loci or envelopes. Of a binary form, or an ordinary equation in one unknown, one seems to have a satisfactory picture in a set of a definite number of points on a line; but in endeavoring to apprehend even the picture of a plane curve, one begins instinctively to look for inflexional tangents, bitangents, and soon for polars, Hessian, Cayleyan, and other auxiliary loci; so that an algebraic curve becomes to our understanding an associated host of curves, even before we rise to the concept of a class of Riemann surfaces. And in fact, by reflex influence, since the developments of the past forty or fifty years, the geometry of sets of points on a line is now concerned less with the individual set, but much more with the related infinite linear systems or involutions.

Hence simultaneous equations were necessarily to be discussed, if geometry was to develop in the algebraic direction foreshadowed by Descartes and Newton. Descartes it was who proposed to classify plane curves as algebraic and not algebraic, and to divide the former according to the degree of their equations. Analysis quickly overtook pure geometry in developing the properties of loci of the second order, and it was easy for Euler to prove that the equation for the intersection of two second order loci is of degree 4. The next step in generalization was to determine the degree of the eliminant, or equation for the common points satisfying two simultaneous equations of any orders, m and n. This question was solved independently in 1764 by Euler, whose interest in mathematics was universal, and by a young French student who seems to have confined himself exclusively to this narrow field of research, Etienne Bézout. Both gave the degree as $m \cdot n$, the product of the orders of the intersecting loci, and both proved the theorem by reducing the problem to one of elimination from an auxiliary set of linear equations. Both, that is, depended upon the formal structure of what were later named *determinants*. The result of this initial publication, restricted to two equations, is what has kept current the fame of Bézout, and the determinant resulting from his method is what Sylvester and later writers call the Bézoutiant. But this was to Bézout only the beginning of his lifelong study in the formation of eliminants.

The mode of formation of this resultant of two equations in one unknown, or eliminant of two in two variables, is familiar to most students; Brill and Noether show that an equivalent process was employed by Newton for forming resultants of equations of low degrees, though without the almost obvious extension to equations in two unknowns. Two things about it were patent as clues to generalization. First, it gave a direct process for combining any two equations to eliminate any one unknown. Second, the resultant when found is a linear combination of the two original functions, with multipliers that are rational in the variables and in the coefficients of the given functions,

$$R \equiv F_1 f_1 + F_2 f_2.$$

With more equations and more variables, which of these two features would be more useful? We must remember that in 1765 not even the degree of the eliminant was known, and that the chief effect desired in a scheme for removing two or more unknowns from a set of equations was that it should indicate the degree of the result (e. g., the exact number of intersections of three surfaces of given orders).

The use of a direct process, to be applied step by step in some systematic sequence, is certainly the more tempting; and this mode has been employed very extensively of late — the highest common factor process — as affording more cogent deductive arguments at every stage. But it has its difficulties, as Bézout indeed found out; for he experimented many years before he found the more feasible plan. If one determines to use it, as does Kronecker for example, one must discriminate with care between an essential factor and an adventitious factor in every resultant, and learn to calculate in advance the degree of each. Bézout chose finally the other horn of the dilemma, and availed himself of the right to use hypothesis; he ventured to guess that the other style of attack would be successful.

It was his hypothesis that from $k+1$ equations k variables could be eliminated at one stroke; and that the result would be a linear combination of the given functions. This latter is equivalent to the famous Fundamental-Satz of Noether, and its proof is usually based upon the existence and the particular degree of the resultant. So we recognize the venturesome character of this attack. Supposing, then, that by using multipliers F_i of degree sufficiently high he could produce a combination

$$R_n \equiv F_{m_1}f_{n1} + F_{m2}f_{n2} + \cdots + F_{m_k}f_{nk} + F_{m_{k+1}}f_{n_{k+1}}$$

that should be free from k variables or unknowns, say

$$x_1,\ x_2,\ \cdots,\ x_k$$

and contain only x_{k+1}, he set out to determine the necessary value of n, the degree of R in the sole remaining unknown x_{k+1}. That is to say, he sought for a minimum value n which should be sufficiently great for the purpose.

Naturally his work has received many improvements from later workers; and we must recognize the large credit due to Professor Eugen Netto for having worked out so many supplementary lemmas in his admirable Vorlesungen über Algebra. To rescue a theory from its own weaknesses is no less deserving of honor than the invention of new theories. Only by a vast amount of such patient self-forgetful devotion to the interests of science can the body of demonstrated truth grow as a solid and unshakable structure. It will not injuriously misrepresent Bézout therefore if we say that he considered all the coefficients in the multipliers $F_{m_1}, F_{m_2}, \cdots, F_{m_{k+1}}$ as undetermined arbitrary parameters, and inquired first *how many of them must remain arbitrary after the resultant R_n has been completely determined.* Obviously any two terms in the combination could be modified together, since identically

$$F_\alpha f_a + F_\beta f_b \equiv (F_\alpha + \phi_{a,b}\cdot f_b)f_a + (F_\beta - \phi_{a,b}\cdot f_a)f_b,$$

so that the coefficients in all such functions $\phi_{a,b}$ will remain *essentially indeterminate.* But again, in this way certain modifying terms are enumerated more than once (as those containing the product $f_a \cdot f_b \cdot f_c$, etc.). Hence was necessary a considerable invention and application of a calculus of finite differences to solve the enumerative problem : this forms the first part of Bézout's great work ; and this is reproduced well by Serret (Algèbre supérieure) and Netto.

Excluding essential indeterminates, call the number of *available* undetermined parameters P, or say $P(n, k+1)$; and let $N(n, k+1)$ denote the number of terms that can occur in a polynomial of order n in $k+1$ variables. Since $n+1$ terms in x_{k+1} are expected in R, the number to be removed is $N(n, k+1) - n$. Hence the inequality to be satisfied, stating that there are no more terms to be excluded than there are available arbitrary multipliers, is

$$P(n, n_1, n_2, \cdots, k+1) - 1 \geqq N(n, k+1) - n - 1$$

or

$$N - P \lesseqgtr n.$$

If R is to be a determinate function, this shows that its degree must be exactly $N - P$. But as n increases from the largest of the numbers $n_1, n_2, \cdots, n_{k+1}$, $N - P$ reaches soon its maximum value and then, as a constant, remains superior in magnitude until n becomes equal to the product $n_1 \cdot n_2 \cdot n_3 \cdots n_{k+1}$. From that point on of course the inequality is reversed ; that is, the parameters are more numerous than the conditions to be satisfied. Hence *the degree of any completely determinate resultant must be exactly the product of the degrees of the original equations.*

This leaves unsettled of course first the question of the *existence* of such a resultant, namely, a determinate linear combination of the given functions with rational multipliers ; in other words the question whether for $n = N - P$ the linear equations to be satisfied are all independent and consistent. Also, secondly, it leaves to be fixed by further consideration the choice of terms whose coefficients shall be equated to zero, in order from the linear equations thus formed to eliminate the auxiliary coefficients by means already familiar and so to obtain the eliminant expressed as a determinant. On this second point no one has yet succeeded in elaborating such a scheme for substitution as evidently Bézout supposed was practicable. We shall find his own words on this point not critically precise, but interesting as showing how loosely constructed arguments once passed current. "But if we imagine that the first equation is multiplied by a complete polynomial of order m in the same number of unknowns, and that in the resulting equation of degree $m_1 + n_1$ (the 'product equation') there are substituted in all the terms where it is possible to do this the value of $x_1^{n_2}$, that of $x_2^{n_3}$, that of $x_3^{n_4}$, etc., then as the multiplier polynomials will have brought into the product equation as many different coefficients as there are terms, we see that after those substitutions there can remain of terms in x_1, x_2, x_3, etc., only so many as it will be possible to make disappear by the help of the coefficients in the polynomial multipliers." Further on, more to the same effect about these substitutions, but nowhere an examination into the difficulties of such substitutions when more than one quantity is to have its exponent reduced.

No such difficulty arises if only a very special set of equations is considered, namely, that in which each equation in its turn contains only the first power of its correspondingly numbered unknown quantity together with other terms involving later unknowns but none earlier,

$$f_{n_1} = x_1 - g_{n_1}(x_2, x_3, x_4, \cdots, x_{k+1}) = 0,$$
$$f_{n_2} = x_2 - g_{n_2}(x_3, x_4, \cdots, x_{k+1}) = 0,$$
$$\cdots\cdots\cdots\cdots$$
$$f_{n_k} = x_k - g_k(x_{k+1}) = 0,$$

while the last may be of any form $f_{n_{k+1}}(x_1, x_2, \cdots, x_k, x_{k+1}) = 0$. What Bézout very properly terms substitution is described more fully by precisians of the present century as the addition of the function f_i multiplied by a suitable polynomial; since a term like ax_1^r for instance is replaced by adding to it the product

$$a(x_1 - g_{n_1}) \cdot \left\{ \frac{x_1^r - g_{n_1}^r}{x_1 - g_{n_1}} \right\} \equiv a(x_1^r - g_{n_1}^r).$$

In this example, an unknown once removed from $f_{n_{k+1}}(x_1, x_2, \cdots, x_k, x_{k+1})$ remains absent, and the first k of them being removed thus from the $(k+1)$th, there remains the eliminant in the single unknown x_{k+1} to the proper degree.

But for the general "complete" equations, the attempt to construct the elimination process *progressively* after this model has not met with success hitherto, since an unknown after having its degree lowered by an early substitution may have it raised by a later substitution. Even Netto here attempts nothing further than a demonstration by the aid of linear equations. The general case of course would be not the very simple example cited above; each variable would appear in each to several different powers, and direct substitution would lower the degree of one variable only; the aim being to depress the degree to a maximum $n_r - 1$ for x_r $(r < k + 1)$. If this is possible, it gives what may be called for certain purposes a normal form.

Now this is a topic of scarcely less interest and importance than the matter of elimination itself — the reduction of a polynomial, by the aid of a set of equations, to another form in which the variables shall appear with exponents not higher than prescribed limiting values. This is for Bézout a transition form, from which finally the coefficients of all terms containing the variables to be removed are set equal to zero, giving the linear system needed for the closing step of his process of elimination. But as the form is unique, it should prove and has proved equally interesting, as a reduced normal form in a modular system, with the other reduced normal form that we call *the* eliminant.

There were still questions to be investigated when Bézout laid down his pen and published his *magnum opus* in 1779; but certainly he did not leave the subject where he found it. Where he left it, Cayley was to resume it later and at one stroke express a resultant not indeed by a single determinant, but by a fraction whose numerator and denominator contain only determinant factors, found rationally from the coefficients of the given equations. Bézout's system of equations is made to yield a definitive result when once the distinction of available and non-available parameters is abrogated, and all are treated alike. And as for the *degree* of the eliminant, the main problem in his time, Bézout had fixed that both for the so-called general case, and also for a large number of particular cases. One may say that he determined the number of finite intersections of algebraic loci, not only when all the intersections are finite, but also when singular points, or singular lines, planes, etc., at infinity occasion the withdrawal to infinity of certain of the intersection points; and this at a time when the nature of such singularities had not been developed.

All algebraic geometry is a reduction, actual or possible, of systems of equations to a different form, plus the specification of what the symbols shall denote. The resultant being once known, of course its interpretation will thereafter be found often in geometric discussions. To survey the applications of the eliminant therefore is practically to enumerate the different chapters that have been created in algebraic geometry.

The first would certainly be what is called Cramer's paradox. This antedates Bézout, for Cramer's book appeared in 1750, and may probably have helped to form Bézout's opinion on the urgent need for a solution of his problem. The paradox is, briefly, this. The equation of a plane curve of order n contains $\frac{1}{2}(n + 1)(n + 2)$ terms, depends therefore on the same number of coefficients, less one. To require the curve to pass through that number of given points is then, algebraically speaking, to determine the coefficients by means of linear equations, hence *uniquely*. But two curves of that order have in common more than that number of points, namely, n^2 (according to the Bézout theorem). But $n^2 > \frac{1}{2}(n^2 + 3n)$ if $n > 3$, so that two curves, indeed an infinite pencil of curves of order n can be passed through even more points than would commonly suffice to determine a single curve. This not intricate puzzle was resolved by Euler, leading to the interesting and important notion of interdependent sets of points in a plane. From this concept was to arise later that of involutions of points in a plane (or in space of more than two dimensions), and thus was opened a field which has only been entered, but not yet explored. An involution is generated when in a set of points of intersection of two (or three) curves enough points are fixed to leave only one still arbitrarily variable, while $r - 1$ others vary with it. This one then being supposed to describe the whole plane, the totality of positions of the whole set of r varying points, a doubly infinite system of point sets, constitutes an *involution* in the plane.

Next in order of development is the notion of multiple points of curves. Bézout treated them by implication only, locating them at infinity, and showed for many cases how they affect the number of finite intersections. These questions were taken up, five decades later, by Plücker, and led to the discovery of the universally known Plücker relations that connect the numbers of double points, inflexional points, double tangents, and cusps, of a plane curve. These latter gave to geometry the notion of the *deficiency* of a curve, that notion which was to be analyzed to its depths by Riemann and by Clebsch, and to form the chief organic bond between algebraic curves and the theory of multiply periodic functions of a complex variable.

But Cramer's paradox led naturally also to the examination of the sets of variable intersections cut out upon a curve of one order by a varying curve of different order; that is, to the questions of curves having contacts one or many on the fixed curve; of the degree of freedom in a variable point set on the curve; in short to all those questions which lead through Noether's Fundamental-theorem. Before Noether there were theorems in plenty upon the partition of the intersections of two curves into two sets, with the hypothesis that one partial set lie all on one curve of lower order, and the conclusion that the remaining partial set must lie on another curve of complementary order. The best known of these are ascribed to Plücker, Jacobi, and Cayley. But Noether was the first to formulate precisely and prove what was tacitly assumed in them all, that a curve containing all the intersections of two others, $f_1 = 0$ and $f_2 = 0$, can always be represented in the form $F_1 f_1 + F_2 f_2 = 0$, where F_1 and F_2 are rational in the coordinates. One sees that this form is that in which Bézout assumed the resultant, and which he in effect established. Very naturally therefore Noether's proof begins upon that basis, and lays down conditions which while not demonstrably *necessary*, are certainly *sufficient* for the conclusion. Upon this basis then rises the extensive body of theorems relating to adjoint and non-adjoint curves, and in particular the elegant theorem on residual and corresidual point sets on a curve: *If a point set A is corresidual with another B in respect to any third set C, then any fourth set residual to A is residual also to B;* two residual sets constituting, together with the singular points of the basal curve, its complete intersection with an adjoint curve.

Bézout's discussions had regard altogether to the elimination from numerical equations; for in his day the theory of forms, of invariants under the projective group, was not yet known. When that theory arose, the interest of algebra in the resultant was no less keen than that of geometry in the eliminant. Hence in the plane the result of eliminating both coordinates from three equations was examined, an invariant whose vanishing gives the condition for three curves to have a point in common. But three curves might have two, or three, or more, points in common. Cayley therefore proposed the question: If all the first polars of a curve have d points in common, what will be the degree, in the coefficients, of the condition for the occurrence of one further intersection? His answer was $3(n-1)^2 - 7d$, n

being the order of the given curve. More general was Brill's inquiry: When three curves of orders n_1, n_2, n_3 have d points of intersection, what is the condition for the occurrence of one further common point? This question he answers; and so was instituted the theory, yet in its infancy, of *reduced discriminants* and *reduced resultants*.

Pausing briefly in this review of the potentialities of the theorem on the degree of the resultant, though we have spoken so far only of resultants of two forms, let us observe Bézout's own view of the importance of his investigations. In the dedication he says of the work: "It has as its aim the completion of one part of the science of mathematics, on which all the other parts are waiting for that which can at this time secure their own advancement." Then in his preface, after deriding those who had been discouraged from researches barely begun by the complexity of the algebraic relations which they encountered, he continues as follows. "The analysis of infinitesimals, equally attractive and important, . . . has drawn away all the interest and all the toil, and the algebraic analysis of finite quantities, to start from that epoch, seems to have been looked at only as a field in which either there remained nothing further to be done, or else whatever was left to do would have proved fruitless speculation. . . . If we note carefully the fact that in reference to the countless number of equations and of unknowns, upon which the solution of any problem may depend, we know as yet how to treat only the case of two equations and two unknowns; that we understand, I repeat, how to treat only that single case with the certainty of introducing nothing extraneous to the question, then we shall doubtless agree that in this matter everything is yet to be done."

After explaining his methods in 463 pages, Bézout sums up the outcome thus (page xxi). "We think it possible to state that there is no kind of algebraic equations for which we have not given the means of determining the lowest possible degree of the final equation; either when there are or when there are not relations between the coefficients which could occasion a particular lowering of that degree." This belief may be a trifle optimistic, at least it is to be hoped that there will be found means of reaching *enumerative* results with less labor than Bézout's methods would impose. But surely the aim was high, and worthy the devotion of one life time; and the conclusion of his preface must certainly excite our admiration and respect.

"We hope that this work may prove the occasion of great progress in analysis, by turning toward that important field the talent and the cleverness of analysts of our time. We shall regard ourselves as fortunate if on considering the point where we take up these questions and the point where we leave them, it shall be found that we have discharged some part of that tribute which every man owes to society in that state in which he happens to be placed."

A life of unremitting labor is not ill spent if it leaves a work so easily intelligible, so full of interesting problems, and in proportion to contemporary science so complete as this Théorie générale des équations algébriques of Bézout. Yet what a commentary on the futility of the best efforts is found in the fact that both Jacobi and Minding, only 60 years later, published investigations as new whose methods and results were in effect identical with Bézout's! At least this showed not that his work was unnecessary, but only that he was in advance of his time. Was it perhaps that geometry was waiting for Gauss to prove that the degree of an equation indicates actual and not illusory roots? Or was it waiting for Monge and Poncelet and Plücker to set it free from the restraints imposed by a particular system of coordinates? Or perhaps for Liouville or Poisson to adjoin the linear function of all the coordinates with as many indeterminates: $z = k_1x_1 + k_2x_2 + \cdots + k_{k+1}x_{k+1}$, and to discuss not the resultant or eliminant in a single x_i, but the eliminant in z containing all the indeterminates $k_1, k_2, \cdots, k_{k+1}$? The algebraic projective geometry of the present has grown and flourished upon all these preparations; and it appears that not the single isolated worker but the cooperation of a large number, with the utmost facility of communication of results, is needful for the rapid advancement of any branch of mathematics.

To return to the geometrical problems that grow out of the theory of eliminants. In three dimensions, three surfaces intersect in points, real or imaginary, whose number is the product of their three orders. Three quadric surfaces meet in 8 points. Here, as in the plane, arises the question of the interdependence of these points: Cramer's paradox is applicable to space of any number of dimensions, and indeed to intersection systems which are not all points. Of these 8 intersections of three quadrics, if seven are given the eighth can be constructed linearly; and it is surprising to see how many eminent geometers have found in this one problem points of difficulty worthy of their attention. But an eliminant in more than two dimensions, *i. e.*, an intersection system of three or more loci, has a further peculiarly interesting possibility.

Three surfaces may have not only a finite set of points in common, but equally well also a whole curve or system of curves. That is to say, their eliminant may vanish identically, having an infinite number of roots. Three quadric surfaces can have in common a line, two lines, a conic, three lines, or a twisted cubic curve. What would then be the number of additional discrete points of intersection? If there is a common conic C, for example, two of the surfaces meet in C and a second conic K, which have two points in common; and the third surface must cut K in 4 points, the two upon C, and therefore only *two* outside the common conic C. Or speaking briefly, a common conic lying on three quadric surfaces *absorbs* six of their eight intersections. Similarly, a common line absorbs 4, a twisted cubic curve all 8, of the points of intersection of three quadric surfaces. This line of problems is not without difficulty, and seems not to have come to the notice of Bézout except where the common curves were straight lines at infinity; and even here he does not put the description into geometric language.

Evidently there was need of a theory of twisted curves in three dimensions; of gauche surfaces and curves in four dimensions, etc. Cases were known in which not 3, but 4, surfaces were required to define an intersection curve. For this case however there was in Bézout's eliminant a suggestion of a useful mode of attack. By eliminating one variable from two equations one has as eliminant the equation of a cone with vertex at infinity. Treat this in homogeneous coordinates, and we have a cone with vertex at any point, and containing the curve in such a way that on each generator there lies a single point of the curve. A different way of conducting the elimination suggested to Cayley the surface called, since his invention, a monoid; and both Noether and Halphen found the cone-and-monoid a sufficient representative of any twisted curve in 3 dimensions. But of curves and surfaces in projective space of four or more dimensions, I believe that no one has yet worked out a systematic list.

It is to be expected that fundamental propositions will gradually become obscured by their derivatives. When once a theorem like Noether's has been discovered, which embodies the geometrically most available substance of the Bézout theorem, and when this has found expression in geometric propositions on point sets, residues, and groups of point sets on a curve, then it will be but infrequently that geometers will have recourse to the formation of resultants or eliminants. Yet it is not well that such a basal theorem together with the scheme of operation that it involves should again fall into oblivion. For even at this late date one may read in a quite recently issued American cyclopedia the statement that, in complicated equations, elimination becomes difficult and often impossible. And even in Cayley's article in the Encyclopædia Britannica one reads the more temperate verdict that there does not yet exist a distinct theory of systems of equations ! Such statements in what purport to be standard popular works of reference are more often read and credited than the precise summaries given in technical works or even the new mathematical Encyklopädie.

Two relatively recent essays employing resultants and discriminants for the extension of the theory of curves are by Franz Meyer, in the *Mathematische Annalen*, volumes 38 and 43. They proceed from the postulate that truths purely numerical can be established for the kind of curves called rational, and will then hold true for non-rational curves of the same order. The first deals with the ordinary singularities of plane curves; the second, with those of twisted curves in space, determining what consequences ensue to other singularities when two of any one sort come to coincide: *e. g.*, how many inflexions come together as two contacts of a double tangent merge into a hyperosculation point. The immediate aim is to find relations among the real singularities as distinguished from the imaginary ones; but the postulate seems valid and susceptible of adaptation to other uses.

It is not proper to close this sketch without alluding to Kronecker's work in formulating a systematic theory of systems of equations. Under the head of *Modular Systems* he instituted inquiries far beyond the single question of eliminants which filled Bézout's horizon. Some time ago allusion was made to one reduced form of equal significance with the eliminant, where all the variables were retained, but with exponents so lowered that the reduced form was uniquely determined. The study of all such reduced forms under any given system of equations (or moduli), whether general or special, is part of Kronecker's programme, and in particular the grouping together of all forms, which, like the resultant, are reducible to zero by the aid of given equations, under the class name of an algebraic modulus. In his Festschrift and the later expository papers of his pupils are proposed methods for testing any given system for its character, whether general, or special of the first sort (loci with a curve in common), or of the second or higher sort (loci with a surface, etc., in common). The expansion of this body of doctrine or abstract theory into a concrete *geometry* with fulness of examples remains a task, not all deductive but largely creative, for coming decades or generations.

Not the possession of eliminants actually calculated by Bézout's deservedly famous scheme is needful for the geometer, but the knowledge of the conditions under which such an eliminant will exist, and what conditions will modify it. So with regard to the more far-reaching scheme of Kronecker; it is ultimately, perhaps, not the full elaboration of particular examples as such, that we wish to have, but a precise knowledge of *how* the relative operations could be executed in finite time, and a precise formulation of conditions that would modify or influence the result of those operations. Which is of greater value, the logic or the concrete object to which it is applied ? Let everyone decide when both are in his possession !

ON THE REPRESENTATION OF NUMBERS BY MODULAR FORMS.

BY PROFESSOR L. E. DICKSON.

(Read before the Chicago Section of the American Mathematical Society, January 2, 1909.)

1. FOR any field F in which there is an irreducible equation $f(\rho) = 0$ of degree m, the norm of

$$x_0 + x_1\rho + x_2\rho^2 + \cdots + x_{m-1}\rho^{m-1}$$

is a form of degree m in m variables which vanishes for no set of values x_i in the field F, other than the set in which every $x_i = 0$. For a finite field it seems to be true that every form of degree m in $m + 1$ variables vanishes for values, not all zero, in the field. For $m = 2$ and $m = 3$ this theorem is