$$(18) \qquad \rho(x)D(\lambda) = \int_0^1 \psi(s)F(x, s, \lambda)ds,$$

where $F(x, y, \lambda)$ denotes the limit to which $F_n(x, y, \lambda)$ tends,

$$F(x, y, \lambda) = \lambda\kappa(x, y)$$
$$(19) \qquad + \sum_1^\infty \frac{\lambda^{p+1}}{p!} \int_0^1 \int_0^1 \cdots \int_0^1 \kappa\begin{pmatrix} x, & s_1, & \cdots, & s_p \\ y, & s_1, & \cdots, & s_p \end{pmatrix} ds_1 ds_2 \cdots ds_p.$$

The convergence of this expression can be established by Hadamard's theorem.

If, in equation (18), we replace $\rho(x)$ by its value taken from equation (2), we get Fredholm's solution of equation (1)

$$(20) \qquad \phi(x) = \psi(x) - \frac{1}{D(\lambda)} \int_0^1 F(x, s, \lambda)\psi(s)\, ds.$$

It is to be observed that the above demonstration establishes the uniqueness of the solution for every value of λ for which $D(\lambda) \neq 0$.

NEW YORK,
April 1, 1909.

THE CHICAGO SYMPOSIUM ON MATHEMATICS FOR ENGINEERING STUDENTS.

Symposium on Mathematics for Engineering Students. Being the Proceedings of the Joint Sessions of the Chicago Section of the American Mathematical Society and Section A, Mathematics, and Section D, Mechanical Science and Engineering, of the American Association for the Advancement of Science, held at the University of Chicago December 30 and 31, 1907.

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ENGINEERING education, having passed through an infancy somewhat of the "ugly duckling" type in the academic family, and then an adolescence of growth too rapid for either garments or comfort, seems now to have arrived at a stage of maturity in which it recognizes the necessity for heart-searching as to its own real character and habits of life. The demands upon its energies from all quarters are heavy and insistent. It must maintain its vigor and efficiency by ridding itself of every superfluous ounce of adipose, and by keeping brain, nerves, and muscles in the best condition. Thus the traditional fundamentals of diet and hygiene must be critically scrutinized, and first of all, mathematics.

Mathematics from its antiquity and its necessity has so long been cast into pedagogic moulds that most people cannot separate their notions of it from these moulds — arithmetic, geometry, trigonometry, etc. It has been taught in some form to nearly all civilized persons, and unfortunately, but necessarily, to a great extent by teachers of very slender mathematical endowment — for this endowment is exceedingly rare. Its form and subject matter have been determined by all sorts of considerations of a more or less arbitrary character.

In our own country most of those who took college work in mathematics more than, say, twenty years ago — and this naturally includes most of the participants in the present symposium — were diligently drilled in Todhunter or some dilution of his texts, with little chance of appreciating the real vitality of mathematics or its relations to other living subjects. Those who escaped permanent aversion for it, had by their industry acquired mathematical technique and some power of concentration. This sort of education, while sometimes redeemed by the teacher's personality and doubtless not inferior to much contemporary work in other lines, could not long withstand the influence, on the one hand, of the new generation of mathematicians "made in Germany," and on the other, of competition with the modern scientific or technological subjects. It is safe to say that it hardly now survives to any important extent — except in the minds of certain critics who still describe the conditions of their own youth with fine but unscientific indifference to the progress made in the intervening years. On the other hand, it is but just to grant that the critical revision of a great body of knowledge, and the training of an army of teachers with new points of view — both indispensable steps — have not been matters for a day or a decade. The mere substitution of engineers to teach mathematics would have been vastly simpler, but of course utterly inadequate, not to say futile, as a remedy.

In the accomplishment of this piece of educational evolution, cooperation has been, and will continue to be, essential, and it is as a welcome form of such cooperation that the Chicago symposium, which is here reported, has its chief significance.

The report opens with a valuable description of the "present condition of mathematical instruction for engineers in American colleges," based on a study of seventeen selected institutions, by Professor Townsend, of Illinois. There is a judicious deprecation of undue advance of entrance requirements in mathematics, and of undue emphasis on uniformity. The statistical tables of entrance requirements and of time allotment in course are naturally to be used only with caution for purposes of comparison, and with due regard for Poincaré's dictum, that mathematics — a fortiori statistics — is the art of calling different things by the same name. Interesting references are made to the desirability of elective mathematics, to the positive and negative value of the "Perry movement," to recent tendencies to deal with mathematics as one subject rather than as many, etc. It is wisely urged that what is best for the engineering student in the first two years of his mathematical work is also best for the student who is taking mathematics as an element in a liberal education. In view of the emphasis put by some other speakers on the use of mathematics as a "tool," it may be pertinent to supplement the remark just quoted by adding that the best engineering schools and departments aim to combine in a truly liberal sense general and professional education, and that any attempt to teach mathematics merely as a tool would, to the extent of its dubious success, mean sacrificing the first of these objects. One of the important elements in higher — as distinguished from merely technical — education is surely the student's relation with teachers of widely different points of view, mathematical, philosophical, literary, etc., as well as scientific and engineering. It is perhaps natural that the same critics who see in mathematics nothing but a machine into which data are fed, and from which results are ground out, should most insist that the subject ought to be taught as a tool.

The symposium includes further a paper on conditions in foreign countries (chiefly Germany) by Professor Ziwet of Michigan, general papers by Mr. C. F. Scott of the Westinghouse Company and Dr. R. S. Woodward of the Carnegie Institution, a presentation of the standpoint of the practicing engineer by Mr. Ralph Modjeski of Chicago, and Mr. J. A. L. Waddell of Kansas City; of the standpoint of the professor of engineering by Professors Williams of Michigan, Talbot of Illinois, and Swain of the Massachusetts Institute of Technology; of the standpoint of the professor of mathematics, by Professors Slichter of Wisconsin, and Woods of the Massachusetts Institute of Technology, and by President McNair of the Michigan College of Mines.

For the readers of this BULLETIN it may naturally be most profitable to consider the criticisms of the engineers—including Mr. Scott—and the professors of engineering. Of the three engineers, one who received his mathematical training abroad and naturally expresses himself with some reserve in regard to American conditions, looks forward to the time when only the applied mathematics will be taught in college, all necessary abstract mathematics forming a part of the entrance requirements. The other two dwell on unsatisfactory elements in existing conditions, and on results to be sought, both of them emphasizing the importance of skill in the use of mathematics as a tool. This view, urged still more emphatically by one of the professors of engineering, might remind an unsympathetic reader of the ingenuous evidence so often presented to the Committee on Ways and Means by beneficiaries of the protective tariff. As a matter of course, the mere use of mathematics as a tool can not be most successfully taught by mathematical specialists. Exactly this is, and must always be, one of the most important functions and opportunities of the teachers of physics, mechanics, and engineering, as was naturally insisted in the course of the discussion by the mathematical speakers. The mathematicians must build a foundation adapted to the educational superstructure. With this their responsibility ends. If they permit themselves to be diverted from this responsibility for laying a solid foundation, it needs no engineer to foretell the disastrous consequences.

The further criticism of the engineers and the professors of engineering, while varying widely in details, seems to connect itself with the general notion that mathematics is habitually presented to engineering students in too abstract a form, with too little attention to the statement of problems and the interpretation of results, and with too little appreciation of the uses to be made of mathematics in engineering work. There is still unquestionably a measure of truth in this criticism, but for reasons which have been already indicated, the extent to which the criticism is valid can be determined for each institution only by a careful investigation of what it is now doing. No one of the speakers in question mentions having made such an investigation, even in a single institution. One of the engineers remarks that "possibly methods have changed of late years, but nothing that the writer has seen or heard indicates to him that any fundamental improvement has been effected." One of the professors of engineering states that in his classes he has had students from most of the principal colleges and technical schools of the country, and that he has failed to notice any great difference in them in respect to lack of mathematical power. Such statements hardly suggest that degree of careful discrimination which may be fairly expected of a scientific critic on a scientific subject.

The criticism that teachers of mathematics neglect the applications, so that their students do not learn to *do* anything, suggests the inquiry: Just what do they need to learn to do? In the large engineering institutions and departments the mathematical instruction must often be given to classes made up of students from several courses, or departments, for whom the number and variety of applications of common interest will naturally be limited. Moreover, the stronger engineering schools draw a considerable — and probably an increasing — proportion of their students from academic institutions, in which mathematics has been taught necessarily without specialization. Under these conditions all that can fairly be demanded of the teachers of mathematics is that they articulate their work as closely as may be with the dependent physics and mechanics which immediately succeed it, by including physical and mechanical problems of general interest.

The plea of several speakers that geometry be more emphasized, and that descriptive geometry made more mathematical, deserves careful attention. This question is, of course, to a considerable extent one of organization rather than of mathematical teaching per se. Any thorough-going improvement would seem to imply that descriptive geometry, and perhaps mechanical drawing, should be given under the direction of the mathematical department as its laboratory work, rather than independently, as is now apt to be the case. Undoubtedly teachers of mathematics would welcome any tendency on the part of their colleagues having charge of these outlying subjects to give them a more mathematical character.

It is not without interest that the most severe criticism of the present teaching and teachers of mathematics is coupled with a citation of Sir William Hamilton as an authority on the value and the pedagogy of mathematics. In view of the vogue of Hamilton's criticism, and the limited publicity of Professor Keyser's recent exposé of it, a brief quotation may be made from his Columbia University lecture on Mathematics. In Hamilton's article "the reader is apparently confronted with the assembled opinions of the learned world, and — what is more amazing — they all agree. Literati of every kind, of all nations and every tongue, orators, philosophers, educators, scientific men, ancient and modern, known and unknown, all are made to support Hamilton's claim, and even the most celebrated mathematicians seem eager to declare that the study of mathematics is unworthy of genius and injures the mind. . . . The Scotchman's victory was complete, his fame enhanced, and his alleged judgment regarding a great human interest of which he was ignorant has reigned over the minds of thousands of men who have been either willing or constrained to depend on borrowed estimates. But even all this may be condoned. Jealousy, vanity, parade of learning, may be pardoned even in a philosopher. Hamilton's deadly sin was none of these, it was sinning against the light." It has been shown by Bledsoe and Pringsheim "that Hamilton by studied selections and omissions deliberately and maliciously misrepresented the great authors from whom he quoted — d'Alembert, Blaise Pascal, Descartes and others — distorting their express and unmistakable meaning even to the extent of complete inversion." Even a philosopher thus discredited may have told some truths of his own, but his opinion can hardly be deemed to add much weight to the present criticism. Any attempt to discuss the general educational value of mathematics would, however, far exceed the limits of this review, besides being presumably unnecessary for its readers.

It should not be inferred from any thing that has been said that adverse criticism of mathematics, or mathematicians, was general on the part of the engineering speakers. It is too often the extreme criticism — picturesque rather than accurate — which impresses even a professional public, but the symposium as a whole was rather notable for agreement in fundamentals. As to the importance of the teacher's training and continued interest in applied mathematics, as to the need of emphasis on concrete problems, as to the relatively low value of lectures and mathematical blackboard work, there appears to have been no difference of opinion. It may be added, in connection with the training of teachers of mathematics, that there would be many advantages if a larger share in that training could be taken by the technological schools and departments than is at present the case.

The case for the teachers of mathematics was ably sustained by Professors Slichter and Woods. The former from the standpoint of a consulting engineer as well as that of a teacher of mathematics, emphasizes the increasing need on the part of technology for more and better science, and insists that mathematics is a basal science rather than a tool. Professor Woods dwells particularly on the urgent need that teachers in various subjects dependent on mathematics continue and complete the work which the mathematicians can only have begun. Mention may also be made of the valuable contribution to the subsequent discussion by President Howe of the Case School.

The pamphlet closes with the announcement of a committee of fifteen — the actual list seems to include twenty names — to make to the Society for the Promotion of Engineering Education such a report on mathematics for colleges of engineering as in their opinion will be of service to teachers in such institutions.

After all, however, the real value and importance of such a symposium consists in the intangible influence of the discussion itself on its hearers and readers. Reports and resolutions are at the best merely the stamp on the coin, and too often make but a faint impression thereon. The teaching of mathematics to students of engineering is still in that stage of development by individual initiative in which diffusion of information and circulation of ideas are invaluable, but in which ideas formulated in one year as novel may in the next seem commonplace. Up to a certain point it is easy to agree as to principles; beyond that point efforts at agreement may be for the time quite futile. What is difficult, yet possible and necessary, is a constant improvement of our individual practice. It would be a great service — but an exceedingly difficult one — for any committee or any individual to bring the criticisms of current mathematical teaching to the test of a searching statistical investigation, the only conclusive test. It is also a great but a quite practicable service to prepare for the use of teachers and students of mathematics texts involving more numerous and more varied applications. In this direction the teachers of mathematics have done much and need all support and cooperation — criticism also, if it be constructive, or, at least, discriminating. The need of improvement in line with the constructive criticism is, in the judgment of the present reviewer, freely recognized and continually better met.

H. W. TYLER.

OSGOOD'S CALCULUS.

A First Course in the Differential and Integral Calculus. By WILLIAM F. OSGOOD, PH.D., Professor of Mathematics in Harvard University. New York, the Macmillan Company, 1907, pp. xv + 423. Revised edition, 1909, pp. xv + 462.

PROFESSOR OSGOOD in his presidential address before the AMERICAN MATHEMATICAL SOCIETY * has discussed and illustrated the principles which his experience has led him to consider should govern the teaching of the calculus. In the present text he gives us the detailed application of those principles to the difficult pedagogical problems which confront the instructor in the first course in this subject.

Successful instruction in mathematics requires careful adjustment of the conflicting claims of rigor, formalism, and interest. Professor Osgood has recognized † that rigor is a relative matter particularly in elementary instruction, and has enunciated the principle that in such instruction a discussion is to be regarded as rigorous if it meets all the logical demands which the student can be regarded as capable of appreciating at that time. This principle is at bottom the same as that which governs contemporary judgment of productive work, and its application to instruction is but a recognition of the fact that the mathematical development of the individual differs in general from that of the race at most by a transformation of similarity.‡ It is evident, however, that such a principle must be applied with care, for otherwise it may be cited in defense of a multitude of mathematical sins. If used with judgment, however, as is the case in this text, it becomes the very foundation of successful mathematical teaching.

The applications of this principle are in evidence throughout the book. For example: the theorem on the limit of the sum is at first (page 12) tacitly assumed, then (page 14) mentioned in a footnote, and finally proved (page 15) when the progress

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* BULLETIN, vol. 13 (1907), pp. 449–467.

† *Annals of Mathematics*, ser. 2, vol. 4 (1903), p. 178.

‡ Cf. Cantor, Geschichte der Mathematik, 3 Aufl., Bd. 1, p. 3.