SHORTER NOTICES.

Grundlehren der neueren Zahlentheorie. By PAUL BACHMANN. Leipzig, G. J. Göschen, 1907. Sammlung Schubert, v. 53. 11 + 270 pp.

KLEIN has recently * called attention to the two fundamental types of mathematical development. The one is concerned primarily with the exposition of a given branch of mathematics for its own sake and with its own methods. If this type alone obtained, mathematics would appear as a series of more or less distinct theories — algebra, trigonometry, calculus, etc. — which may show here and there incidental points of contact, but which are not organically connected. The other type of development, on the other hand, is concerned with just this welding together of the various so-called branches into a unified whole. It conceives, for example, the two great divisions of analysis and geometry as being only two different aspects of the same thing. Wherever this second type of development has made itself prominently felt it has meant a gain in power; not simply by giving various interpretations to the same theorem, but, and perhaps chiefly, by making the methods of one theory available for research in another.

All branches of mathematics have felt the influence of this second type of development to a greater or less degree. Even the theory of numbers, which for a while held a certain autonomy in its methods, is at present in its more advanced portions at least in intimate union with other fields, *e. g.*, the theory of functions of a complex variable. In recent years, however, even the elementary portions of the theory of numbers have had to submit to the introduction of matter and methods from another field; viz., that of geometry. To speak of a "modern" treatment of the elementary theory of numbers brings to mind at once the lectures of Klein — that master exponent of the second type of development — during the year 1895–96, and Minkowski's work on the Geometrie der Zahlen. An elementary text on the theory of numbers which makes consistent use of these geometric interpretations and methods where they are available is obviously a great desideratum. It was therefore with much joyful anticipation that the reviewer opened

* Elementarmathematik vom höheren Standpunkte aus, pp. 180–187.

this recent addition to the Sammlung Schubert. The preface seemed to intimate that the author intended to write just such a text as we had hoped for. But after reading the little book with much occasional pleasure the reviewer laid it down with a distinct feeling of disappointment. Of geometric interpretation there is a little; of geometric method there is none.

Having thus expressed our disappointment regarding one feature of the text, it is a pleasure to mention some that are pleasing. The notion of a domain of rationality — or field — is introduced on the first page, and that of a modulus — domain of integrality — on the second. The latter is defined as any set of numbers which is closed under the operations of addition and subtraction; *i. e.*, such that if a and b are numbers of the set, $a + b$ and $a - b$ are also in the set. Thus, if $a, b, c, \ldots$ are any given numbers, the set of all numbers of the form $ax + by + cz + \ldots$, where x, y, z, are any integers, forms a modulus which is denoted by $[a, b, c, \ldots]$. This notion is made fundamental throughout the book; a fact which differentiates it from the ordinary treatment, and which introduces a refreshing unity of conception. Many results are stated in terms of it which are usually stated differently. For example, two integers n, n' are said to be congruent mod m if $n - n'$ is a number of $[m]$. Its advantageous use in the development of Dedekind's theory of ideals is obvious.

The book is divided into two parts devoted respectively to "the rational number field" (157 pages) and to "the quadratic number field" (113 pages). The first four chapters discuss briefly and sometimes with a refreshing novelty the usual topics relating to 1) the divisibility of numbers, 2) congruences, 3) quadratic residues, including the law of reciprocity, 4) the linear form $f = ax + by$. Another unifying principle is found in the fact that the theorem concerning the solvability in integers of the equation $ax + by = 1$ is made fundamental throughout this portion of the text. Chapter 4 contains a complete discussion of the equivalence of moduli $[a, b]$, which is of fundamental importance later in the discussion of ideals. Here also we have for the first time a geometric interpretation of the results by means of a lattice (Zahlengitter). We are sorry that in this chapter the author did not see fit to include also Klein's elegant geometric interpretation of the development of an irrational number into a continued fraction. Chapter 5, which is the last of Part 1, begins the discussion of the equivalence of quadratic forms, the discussion not being completed until later in connection with the theory of quadratic numbers.

The author has developed the theories of quadratic forms and quadratic numbers in conjunction in Part 2, each supplementing the other and forming together a single theory. In this fact he finds his chief reason for calling his treatment "modern." The result is indeed "esthetically satisfying" (quotation from the preface). In view of the fact, however, that the book is professedly intended for beginners, it may perhaps be doubted whether this treatment is pedagogically desirable. There can be no manner of doubt that the beginner will find Part 2 hard reading; and it does not appear evident that the intermingling of forms and ideals, however beautiful the result, makes the reading any less difficult.

The book is remarkably free from typographical errors. Besides the single one noted in the corrigenda, the reviewer has noticed only two; one on page 36, 3d line from below, where *nr.* should read Nr.; and one on page 70, line 7, where the reference should be to Nr. 4 instead of to Nr. 3. In this connection we may note further that on page 25 the expression $\phi(1)$ must be defined as equal to unity, and on page 71 the condition $\Delta \neq 0$ should be added.

J. W. YOUNG.

The Axioms of Descriptive Geometry. By A. N. WHITEHEAD. Cambridge University Press, 1907. viii + 74 pp.

THIS little volume is No. 5 of the Cambridge Tracts in Mathematics and Mathematical Physics. It follows a previous tract (No. 4 of the same series) by the same author, on the Axioms of Projective Geometry, to which constant reference is made. The work begins with formulations of the axioms, those of Peano and Veblen being given in detail. Chapter II treats of the relation of projective space and the associated space obtained by taking a convex region of the projective space, such a convex region being shown to be a descriptive space. The development follows that of Bonola closely. Chapter III contains the development of ideal elements in a descriptive geometry, the work being drawn from Veblen, apparently. In Chapter IV a "General theory of correspondence" is introduced through the medium of projective coordinates, the ideas of continuous groups of projective transformations and their infinitesimal transformations being developed from analytic