correspondence are the following. If the distances between three Clifford parallels are a, b, c, and if the flux-angles between the three rectangular Clifford surfaces joining them are α, β, γ, then these six quantities are related to one another exactly as if their doubles were the sides and angles of a spherical triangle; e. g., $\sin 2a : \sin 2b : \sin 2c = \sin 2\alpha : \sin 2\beta : \sin 2\gamma$. The locus of a line that is right parallel to one of the generators of a cone of order m and left parallel to the others in order is a ruled non-developable surface S of order $2m$. Any curve drawn on the surface S and everywhere orthogonal to its generators will meet any generator in points whose distance apart is constant for the given surface and proportional to its volume.

F. N. COLE,
Secretary.

THE WINTER MEETING OF THE CHICAGO SECTION.

THE twenty-sixth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Friday and Saturday, December 31, 1909–January 1, 1910. Professor G. A. Miller, Chairman of the Section, presided at all of the sessions except at the opening on Friday morning, when Professor E. B. Van Vleck, Vice-President of the Society, occupied the Chair. The attendance at the various sessions included sixty-one persons among whom were the following forty-seven members of the Society:

Professor C. H. Ashton, Mr. W. H. Bates, Professor G. A. Bliss, Professor Oskar Bolza, Professor J. W. Bradshaw, Professor W. H. Bussey, Dr. Thomas Buck, Professor D. F. Campbell, Professor D. R. Curtiss, Professor J. F. Downey, Dr. Arnold Dresden, Mr. E. B. Escott, Mr. Meyer Gaba, Professor E. D. Grant, Mr. T. H. Hildebrandt, Professor F. H. Hodge, Professor T. F. Holgate, Professor Kurt Laves, Dr. A. C. Lunn, Mr. E. J. Miles, Dr. W. D. MacMillan, Dr. H. F. MacNeish, Professor G. A. Miller, Professor E. H. Moore, Professor C. N. Moore, Dr. R. L. Moore, Professor J. C. Morehead, Professor F. R. Moulton, Professor Alexander Pell, Mrs. Anna J. Pell, Miss Ida M. Schottenfels, Professor G. A. Scott, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Professor A. W. Smith, Professor A. L. Underhill, Professor E. B. Van

Vleck, Dr. G. E. Wahlin, Professor E. J. Wilczynski, Professor R. E. Wilson, Professor B. F. Yanney, Professor A. E. Young, Professor J. W. Young, Professor J. W. A. Young.

On Friday evening forty members of the Society dined together at the Quadrangle Club, and enjoyed one of the most interesting occasions in the history of the Section, in the way of social intercourse and the promotion of acquaintance and good fellowship.

At the business meeting on Saturday morning the following officers of the Section were elected for the ensuing year: Professor L. E. Dickson, Chairman, Professor H. E. Slaught, Secretary, and Professor W. B. Ford, third member of the program committee. At this time also the following resolution was introduced by Professor Van Vleck and carried by unanimous vote: Resolved that the Chicago Section respectfully requests the Council of the Society to arrange, in accord with the powers granted to it in By-Laws III and VII, that the next annual meeting of the Society be held with the Chicago Section, and that the President's address be also there given.

The following papers were read at this meeting:

(1) Professor C. H. ASHTON: "A new elliptic function."

(2) Professor C. N. MOORE: "On the uniform summability of the developments in Bessel functions of order zero."

(3) Miss HAZEL H. MACGREGOR: "Three-dimensional chains and a classification of the collineations in space."

(4) Mr. E. B. ESCOTT: "Logarithmic series."

(5) Mr. E. B. ESCOTT: "Calculation of logarithms."

(6) Professor E. J. WILCZYNSKI: "On the problem of three bodies."

(7) Dr. A. C. LUNN: "An abstract definition of limit."

(8) Dr. A. C. LUNN: "Note on the existence of the instantaneous axis in a rigid body.

(9) Mrs. ANNA J. PELL: "On an integral equation with an adjoined condition."

(10) Professor G. R. DEAN: "Generalized plane stress."

(11) Mr. W. H. BATES: "The medium curvatures of R_n in S_{n+1}."

(12) Dr. G. E. WAHLIN: "On the base of a relative field, with an application to the composition of fields."

(13) Professor D. R. CURTISS: "Note on a method of determining the number of real branches of implicit functions in the neighborhood of a multiple point."

(14) Professor E. B. VAN VLECK: "A functional equation for the sine."

(15) Professor E. B. VAN VLECK: "On certain extensions of Abel's functional equations and their relation to Weierstrass's algebraic addition theorem."

(16) Professor G. A. MILLER: "Groups generated by two operators each of which is transformed into a power of itself by the other."

(17) Professor W. A. MANNING: "The limit of the degree of primitive groups."

(18) Professor C. H. SISAM: "On three-spreads satisfying four or more linear partial differential equations of the second order."

(19) Dr. L. I. NEIKIRK: "Groups of rational fractional transformations in a general field."

(20) Professor J. W. YOUNG: "On the discontinuous zeta groups defined by the rational normal curves in a space of n dimensions."

(21) Dr. R. L. BÖRGER: "On the Galois group of the reciprocal sextic equation."

(22) Professor F. R. MOULTON: "The singularities of the solution of the two-body problem for real initial conditions."

(23) Professor J. B. SHAW: "On hamiltonian products."

(24) Mr. A. R. SCHWEITZER: "On the geometry of the projective line."

(25) Mr. A. R. SCHWEITZER: "On the dimensional extension of Grassmann's extensive algebra" (preliminary report).

The papers of Dr. Neikirk and Miss MacGregor were presented by Professor J. W. Young. In the absence of the authors the papers of Professors Dean, Manning, and Shaw, Dr. Börger, and Mr. Schweitzer were read by title.

Besides the above papers, informal reports from two committees of the International Commission on the teaching of mathematics were presented, one by Professor D. R. Curtiss on "Courses of instruction in universities," and one by Professor E. B. Van Vleck on "Preparation of instructors for colleges and universities." Considerable interesting discussion followed these reports.

Abstracts of the formal papers follow below. The numbering corresponds to that of the titles in the list above.

1. In Professor Ashton's paper, a set of four functions $O(z)$

is defined by the functional equation

$$O(z) = [1 - ae^{\frac{2\pi i}{w_1}(z+w_2)}]\, O(z + w_2),$$

under the condition that

$$\lim_{n=\infty} O(z + nw_2) = 1.$$

From this definition the developments of the function $O(z)$ as an infinite product and as an infinite series are determined and some of its properties are studied. Its development in another infinite product by means of Weierstrass's theorem is also obtained and from this it is simply expressed as an infinite product of gamma functions. Its multiplication theorems are determined. Its relation to the σ or θ function is shown to be similar to the relation of the reciprocal of the gamma function to the sine, and finally it is shown that all of the elliptic functions and the constants which occur in a discussion of these functions can be expressed very simply in terms of O functions.

2. In this paper Professor Moore establishes the following theorem : If the function $f(x)$ is finite and integrable in the interval $c \leqq x \leqq 1$, and has a derivative that is finite and integrable in the interval $0 \leqq x \leqq c$, where c is any positive constant < 1, then the development of $f(x)$ in Bessel functions of order zero will be uniformly summable to the value of $f(x)$ throughout the interval $0 \leqq x < x_0$, where x_0 is any positive constant $< c$.

It has already been shown by Professor Moore in a previous paper * that if $f(x)$ is finite and integrable in the interval $0 \leqq x \leqq 1$, the development will be uniformly summable to $f(x)$ in any closed interval lying in the interval $0 \leqq x < 1$, which does not include a point of discontinuity of the function and does not include the origin. The behavior of the series in the neighborhood of the origin was not discussed in the previous paper.

3. The classification of the collineations in space, on the assumption that two collineations are equivalent if one can be transformed into the other by a linear homogeneous transformation with complex coefficients, is well-known. If, however,

* *Transactions Amer. Math. Society*, vol. 10 (1909), pp. 391–435.

the coefficients are restricted to real numbers, the number of types of collineations will be increased. This classification brings up the important additional problem as to the conditions under which a collineation with complex coefficients can be transformed into one with real coefficients.

In a recent paper * Professor J. W. Young has considered these two problems for the complex line from the point of view of projective geometry, the notion of a linear chain being fundamental. In a subsequent paper,† by making use of the idea of a two-dimensional chain, Professor Young considered these problems for the complex plane.

Miss MacGregor applies the same principle of classification to the non-singular collineations in a complex space of three dimensions and considers the corresponding problems. She gives the classification of the collineations in space into nineteen distinct types, each of which leaves a three-dimensional chain invariant. Any such collineation may be represented with real coefficients. The necessary and sufficient conditions that a collineation be of this type are derived and the corresponding systems of invariant chains are determined.

4. The first paper by Mr. Escott completes the one read by him in April, 1904, before the Chicago Section. In the ordinary logarithmic series

$$\log \frac{X+d}{X-d} = 2\left[\frac{d}{X} + \frac{1}{3}\left(\frac{d}{X}\right)^3 + \cdots\right]$$

the problem is to express X in the form of polynomials in x so that both $X + d$ and $X - d$ shall have rational linear factors. The solution of this problem gives interesting applications of the elementary theory of numbers. A number of series of this kind have been developed by others, but there has been no systematic attempt to solve the problem. Mr. Escott shows how an indefinite number of series may be obtained where X is of degree 1 to 7, and gives four examples where X is of degree 10, some of the factors in the latter case being quadratic.

5. In his second paper Mr. Escott shows the application of the series in the preceding paper to the computation of loga-

* "The geometry of chains on a complex line," *Annals of Mathematics*, vol. 11 (1909), pp. 33–48.

† Read before the Society at its April meeting, 1908.

rithms. Huyghens has given a list of numbers differing by unity, having small factors, which may be used to calculate the logarithms of numbers as far as 100. This table is improved and extended to 200.

6. The problem of three bodies has been almost exclusively studied from the analytical point of view. In the present paper, Professor Wilczynski formulates a number of questions suggested by geometry, some of which are capable of direct and final answers, while others of a more difficult character are merely indicated. It is intended that all of these problems be subjected to a more thorough investigation in the future.

The center of mass of the three bodies is supposed to be at rest. Each of the masses will describe a certain curve, the straight lines joining them in pairs will generate three ruled surfaces, the plane of the three masses will envelop a cone. In this first paper the linear differential equations are set up which characterize the projective differential properties of some of these loci. Particular attention is paid to the question : can the ruled surface generated by the straight line joining two of the bodies be developable? It is found that in order that this may be so, the mutual distances must satisfy a certain differential equation of the second order, but the question of the compatibility of this equation with those of the problem of three bodies is provisionally left open. Can this developable be a cone? Leaving aside the trivial cases in which the cone degenerates into a plane or a straight line, whose existence is obvious, it turns out that the vertex of the cone must be at infinity, i. e., the cone is necessarily a cylinder. Moreover, the triangle formed by the three bodies must in that case constantly remain isosceles, the two equal sides being those which join the two bodies which are situated upon the same element of the cylinder to the third. It is then shown that there actually exist solutions of the problem of three bodies for which the triangle always remains isosceles, and that some of these give rise to the "cylindrical solutions" just indicated. The third body, in the case of such a cylindrical solution, always describes a plane curve similar to the corresponding plane section of the cylinder. It does not appear that the plane section of the cylinder can be determined in general as a simple curve, but the differential equations which characterize it projectively are obtained.

Another question which receives a simple answer is that as to the existence of solutions in which one of the bodies describes an asymptotic curve upon one of the ruled surfaces generated by the line which joins it to one of the other bodies. The criteria for plane orbits are also indicated.

It is proposed to found a new general theory of perturbations upon a combination of the theory of osculating conics, cubics, and quartics of plane and space curves, with the method of the variation of constants.

7. In his first paper Dr. Lunn gives a general abstract setting of the concept of limit, designed so as to exhibit as particular cases the ordinary special limiting processes of analysis. The dependent variable is assumed to be numerical, real or n-dimensional complex, but the independent variable is taken as an abstract set of elements, undefined except as it is subject to an order relation satisfying certain postulates. The abstract theory, based on the abstract analogue of the proposition of Du Bois-Reymond, is devoted primarily to several theorems relating to necessary and sufficient conditions for interchange of two limiting processes. These yield as special cases such theorems as those on the integration and differentiation of infinite series, the differentiation of a definite integral, and the behavior of the solutions of differential equations as functions of a parameter.

8. In his second paper Dr. Lunn defines a rigid body with fixed origin as a non-coplanar set of points each at a constant distance from the origin and such that the scalar product of the vectors from the origin to any two points is constant. A purely analytic vector proof is given that there exists a single vector w such that the velocity of any point is given by $v = w \times r$.

9. Mrs. Pell shows that the solution of an orthogonal integral equation with an adjoined condition depends on the solution of an orthogonal equation without any condition.

10. In the case of an elastic plate under the action of a system of forces parallel to the faces and uniformly distributed across the edges, Professor Dean finds that the cubical expansion is a linear function of the two coordinates parallel to the faces; the coefficients being arbitrary constants which are determined by means of given surface tractions or displacements. This function when introduced into the equations of equilibrium

in terms of the displacements reduces the system to three independent equations, two of which are of Poisson's form in two dimensions, the third having a single term and being satisfied by the linear function which expresses the cubical expansion.

The solutions of the Poisson equations are expressed as surface and line integrals. The results substituted in the generalized form of Hooke's law give the normal components of stress and the components of shear.

Using Hertz's method of finding the pressure between two bodies in contact, thus determining the surface tractions and displacements, we have a complete and workable solution of the problem of determining the stresses in riveted and pin joints and in other problems of technical mechanics.

Errors frequently made in applying the theory of elasticity to technical mechanics, chiefly in assigning boundary conditions, are pointed out and means of avoiding them are suggested.

11. At any point of a surface in ordinary space, the curvatures of the two lines of curvature are the roots ρ_1 and ρ_2 of the equation

$$\begin{vmatrix} D + E\rho & D' + F\rho \\ D' + F\rho & D'' + G\rho \end{vmatrix} = 0 .$$

After division by the coefficient of ρ^2, the absolute term of this equation is the gaussian curvature of the surface, and the coefficient of ρ its medium curvature.

Similarly, at any point of a hypersurface R_n in the euclidean space S_{n+1}, the curvatures of the n lines of curvature are the roots $\rho_1, \cdots, \rho_n$ of the equation

$$(1)\qquad \begin{vmatrix} \alpha_{11} + a_{11}\rho & \alpha_{12} + a_{12}\rho \cdots \alpha_{1n} + a_{1n}\rho \\ \alpha_{21} + a_{21}\rho & \alpha_{22} + a_{22}\rho \cdots \alpha_{2n} + a_{2n}\rho \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \alpha_{n1} + a_{n1}\rho & \alpha_{n2} + a_{n2}\rho \cdots \alpha_{nn} + a_{nn}\rho \end{vmatrix} = 0,$$

in which the a's are the first fundamental quantities and the α's the second.

Equation (1) is of the nth degree in ρ and may be written

$$(2)\qquad \rho^n + K_1\rho^{n-1} + \cdots + K_{n-1}\rho + K_n = 0,$$

in which K_n is the Kronecker-Gaussian curvature of the hyperspace, and has been fully treated. In this paper, Mr. Bates studies the other coefficients of (2), which are called the medium curvatures of R_n in S_{n+1}.

12. If K is an algebraic field of degree N and k a subfield of degree n, Dr. Wahlin's paper shows that there exist in K $N/n = r$ numbers $\Omega_i (i = 1, 2, \cdots, r)$, and in k r ideals O_i, such that every integer in K can be expressed in the form

$$\sum_{i=1}^{N} \mu_i \Omega_i ,$$

where the μ_i are integers of O_i, and moreover all numbers so represented are integers in K. It is then shown that the relative discriminant of K is equal to

$$O_1^2 \cdot O_2^2 \cdots O_r^2 \cdot | \Omega_i^{(k)} |^2,$$

where the index k is used to designate the relative conjugates to Ω_i.

These results are then used to deduce an expression for the discriminant of a field compounded from two fields in the case where the degree of the compounded field is equal to the product of the degrees of the two fields divided by the degree of the greatest subfield common to these two fields.

13. In a paper read before the Society in April, 1908, Professor Curtiss developed a method for determining the number of real branches of an implicit function $f(x, y) = 0$ in the neighborhood of a multiple point. It was there shown that if the point (x_0, y_0) is of multiplicity n, and if $\partial^n f(x_0, y_0)/\partial y^n \neq 0$, then the number of real branches passing through that point depends on the number of changes of sign in the values of $f(x, y)$ computed on each real branch of the implicit function $\partial f(x, y)/\partial y = 0$ in the neighborhood of (x_0, y_0). In the present note it is shown that if a_2 and b_2 have any but a certain finite number of values the implicit function $a_2 f(x) + b_2 f(y) = 0$ can be used in place of $f(y) = 0$, the restriction $\partial^n f(x_0, y_0)/\partial y^n \neq 0$ being thereby removed and other simplifications effected.

14. Cauchy has shown that the only real continuous solutions of the functional equation

$$\phi(x + y) + \phi(x - y) = 2\phi(x)\phi(y)$$

(other than the trivial solutions $\phi(x) \equiv 0$ and $\phi(x) \equiv 1$) are $\sin cx$ and $\sinh cx$. Professor Van Vleck gives in his paper a functional equation defining *uniquely* the sine function, and from this equation the properties of the function follow with great ease and rapidity.

15. In one of his earliest published papers Abel showed that if a differentiable function $f(x, y)$ possesses the property that $f[x, f(y, z)]$ is symmetric in x, y, z, then there exists a function $\phi(x)$ such that

$$\phi(f[x, y]) = \phi(x) + \phi(y). \tag{1}$$

Professor Van Vleck extends the theorem to a class of multiform functions $f(x, y)$ and derives a necessary and sufficient condition that an algebraic equation $G[\phi(x + y), \phi(x), \phi(y)] = 0$ shall present an actual (and not an impossible) addition theorem. Two generalizations of (1) were also discussed, the first of which has the form

$$\phi(f[x, y]) = f_1[\phi(x), \phi(y)].$$

16. Two special cases of the groups generated by two operators each of which is transformed into a power of itself by the square of the other have been considered in earlier papers by Professor Miller; namely, when the square of each of the two generators transforms the other generator either into itself or into its inverse. The object of the present paper is to obtain some fundamental theorems relating to the general case where the two operators s_1, s_2 satisfy the conditions

$$s_1^{-2}s_2s_1^2 = s_2^\alpha, \quad s_2^{-2}s_1s_2^2 = s_1^\beta.$$

If at least one of the two numbers α, β is even, the corresponding operator is of odd order and hence it must be generated by its square. In this case the group G generated by s_1, s_2 may be generated by a cyclic group and an operator transforming this cyclic group into itself. As many properties of these groups are well known, Professor Miller confines his attention to the cases in which both α and β are odd.

After observing that the subgroup H generated by s_1^2, s_2^2 is invariant under G and that $s_1^{2(\beta-1)}, s_2^{2(\alpha-1)}$ are invariant operators under G, it is proved that the orders of s_1, s_2 must divide $2(\alpha - 1)(\beta - 1)$. Hence each of these orders has an upper limit whenever α, β are both different from unity and only then. It is proved that the fourth derived of G is unity and hence G is always solvable. It is also proved that the orders of s_1, s_2 are divisors of $2(\beta - 1)^2$, $2(\alpha - 1)^2$ respectively. The general theorems are illustrated by means of the special categories of groups which result when $\alpha = \beta = 1$ and when $\alpha = \beta = 5$. In the latter case the first derived group is abelian.

17. In *Liouville's Journal* for 1871 Jordan demonstrated the fundamental theorem :

"If a primitive group G (not containing the alternating group) contains a substitution A which displaces only m letters, the degree of G cannot exceed a certain limit."

It is a matter of great interest to reduce this limit as much as possible. In two subsequent papers* Jordan has given us expressions for the limit which are much lower than that first published. A further reduction † was recently made by Professor Manning, and the paper presented at this meeting gives a still lower limit. A special case of his general theorem may be stated thus :

The degree of a simply transitive primitive group which contains a substitution of prime order p on q cycles ($q < 2p+3$) cannot exceed the greater of the two numbers $pq+q^2-q$, $2q^2-p^2$.

18. In this paper, Professor Sisam first determines under what conditions a three-spread in space of n dimensions can satisfy more than four linear partial differential equations of the second order. He then shows that if it satisfies four such equations, it has at each point four tangents having contact of the second order with the three-spread. The rest of the paper is devoted to the determination of the conditions under which two or more of these three-point tangents at a generic point may be consecutive.

19. This paper is an extension of that presented to the Chicago Section at the April, 1909, meeting by Dr. Neikirk. Several cases arise and the results of the former paper are extended to include these. The last part of the paper is devoted to finding a transformation $S^{(i)}$ which represents the product of the powers of several substitutions $S^{(i)} = R_1^{y_1^{(i)}} R_2^{y_2^{(i)}} \cdots R_k^{y_k^{(i)}}$.

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* *Bulletin Soc. Math. de France*, vol. 1 ; *Crelle's Journal*, vol. 79.

† BULLETIN, vol. 13 (1907), p. 373.

$S^{(i)}$ is given by a polynomial in x and the exponents $y_r^{(i)}$ $(r = 1, 2, \cdots, k)$.

20. A rational normal curve C_n in a linear space S_n of n dimensions is left invariant by a three parameter group G of collineations $x'_i = \Sigma a_{ij} x_j$ $(i, j = 0, 1, \cdots, n)$ in S_n which is isomorphic with the group of all linear fractional substitutions on a single variable ζ. This isomorphism is most readily effected by interpreting ζ as the parameter of the points on C_n, so that to every collineation of G corresponds a unique ζ-substitution, and conversely. This fact has suggested a method of defining arithmetically certain discontinuous ζ-groups, a problem of the highest importance in the theory of automorphic functions. The method consists in determining G for a given C_n and then considering the discontinuous subgroup of G obtained by restricting the coefficients a_{ij} to be integers with determinant $|a_{ij}| = 1$. If this subgroup contains collineations other than identity, the C_n may be called an *integral* C_n, and the corresponding ζ-group will be discontinuous. That such integral C_n's exist follows at once from the fact that the *canonical* C_n, whose equations are $x_i = \zeta^{n-i}$ $(i = 0, 1, \cdots, n)$ gives rise in this way to the elliptic modular group.*

The case $n = 2$ of this method is the only one which has received detailed treatment.† By the consideration of a certain invariant J it follows that the discontinuous ζ-groups defined by the C_2's cannot contain elliptic substitutions of periods other than 2, 3, 4, and 6. By calculating the corresponding invariant for the general case C_n, Professor Young in a recent paper ‡ showed that the period e of any elliptic substitution occurring in a discontinuous ζ-group defined by a C_n must satisfy the relation

$$\sin \frac{(n+1)\pi}{e} = J \sin \frac{\pi}{e},$$

where J is any integer, positive, negative, or zero. In particular, the values $e = n, n + 1, n + 2$ satisfy this equation for $J = -1, 0, +1$. The present paper is devoted to the proof that integral C_n's really exist whose discontinuous ζ-groups con-

* J. W. Young, "On a class of discontinuous ζ-groups," etc., *Rendiconti del Circolo mat. Palermo*, vol. 23 (1907).

† Fricke-Klein, Automorphe Funktionen, v. 1, p. 533.

‡ BULLETIN, vol. 14 (1908), p. 367.

tain elliptic substitutions of periods n and $n+1$. The method of proof is direct in that it shows how to obtain the equations of a C_n whose group contains a given substitution of period n or $n+1$.

21. In this paper Dr. Börger finds the Galois group of the reciprocal sextic equation

$$x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 = 0$$

for the domain $R(1)$ of the coefficients. The group is a subgroup of the primitive substitution group G_{48}^6 having three systems of imprimitivity. Criteria for the irreducibility of the equation are also found.

22. The differential equations which are satisfied by the relative motion of two bodies subject to the newtonian law of gravitation define, when the initial values of the dependent variables are given, certain analytic functions of the independent variable t. The position and character of the singularities of these functions depend upon the properties of the differential equations and the numerical values of the initial conditions. In Professor Moulton's paper the positions of the singular points are found for all possible real initial conditions, their changes of position are determined as the initial values of the dependent variables are varied, the character of the singularities is found in all cases, the Riemann surfaces are constructed, and (except in a certain degenerate case) a variable is defined as a function of t in terms of which the coordinates can be expanded as power series convergent for all real values of t.

23. Professor Shaw's paper is in abstract as follows: Let there be a system of symbols, $\alpha, \beta, \gamma, \cdots$ either finite or infinite in number. The product of two of these, as α, β, is the symbol $\alpha\beta$, and is of the second rank. The product of $\alpha_1, \cdots, \alpha_n$ is $\alpha_1\alpha_2 \cdots \alpha_n$, and is of rank n. These products may be associative or not, but they are distributive; that is $(l\alpha + m\beta)\gamma = l\alpha\gamma + m\beta\gamma$, etc., where l, m are ordinary numbers. If we restrict the product further we give it a special name and prefix a symbol, as $S \cdot \alpha\beta\gamma$, $V \cdot \alpha\beta\gamma$, from quaternions. These are called partial products.

The first partial product is defined by $I\cdot\alpha\beta$, which is such that it is a scalar, and

$$I\cdot\alpha\beta = I\cdot\beta\alpha\,;\ I\cdot(m\alpha)\beta = mI\cdot\alpha\beta,\ I\cdot(\alpha+\beta)\gamma = I\cdot\alpha\gamma + I\cdot\beta\gamma.$$

Also

$$\begin{aligned} I\cdot\alpha_1\alpha_2\cdots\alpha_{2m} = {} & I\alpha_1\alpha_2\cdot I\cdot\alpha_3\cdots\alpha_{2m} - I\cdot\alpha_1\alpha_3\cdot I\cdot\alpha_2\alpha_4\cdots\alpha_{2m} \\ & + I\cdot\alpha_1\alpha_4 I\cdot\alpha_2\alpha_3\alpha_5\cdots\alpha_{2m}\cdots \\ & +(-1)^i I\cdot\alpha_1\alpha_i\cdot I\cdot\alpha_2\cdots\alpha_{i-1}\alpha_{i+1}\cdots{}_{2m}+\cdots \\ & + \cdot I\cdot\alpha_1\alpha_{2m} I\alpha_2\cdots\alpha_{2m-1}. \end{aligned}$$

These forms turn out to be Pfaffians, and are such that

$$I\cdot\alpha_1\cdots\alpha_{2m} = I\cdot\alpha_2\cdots\alpha_{2m}\alpha_1 = I\cdot\alpha_{2m}\cdots\alpha_1.$$

Next we define the alternate product $A_m\cdot\alpha_1\cdots\alpha_m$, such that $A_m\cdot\alpha_1\cdots\alpha_i\cdots\alpha_j\cdots\alpha_m = -A_m\cdot\alpha_1\cdots\alpha_j\cdots\alpha_i\cdots\alpha_m$. It is of rank m. The Joly products are then defined by the statements

$$A_i\cdot\alpha_2\cdots\alpha_{i+2h} = \Sigma \pm A_i\cdot\alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_i}\cdot I\cdot\alpha_{j+1}\cdots\alpha_{j_{i+2h}},$$

where $j_1 < j_2 \cdots < j_i$ and $j_{i+1} < j_{i+2} \cdots < j_{i+2h}$ and the $i + 2h$ subscripts are permuted in all possible ways, the sign being $+$ or $-$ according to the number of inversions.

Again

$$\begin{aligned} A_{h+i-2c}(A_h\cdot\alpha_1\cdots\alpha_h A_i\cdot\beta_1\cdots\beta_i) = {} & \Sigma \pm A_{h+i-2c}\cdot\alpha_{j_1} \\ & \cdots\alpha_{j_{h-c}}\beta_{k_1}\cdots\beta_{k_{i-c}}\cdot I(A_c\alpha_{j_{h-c+1}}\cdots\alpha_{j_h}A_c\beta_{k_{i-c+1}}\cdots\beta_{k_i}). \end{aligned}$$

(Example: $A_1\cdot A_2\alpha_1\alpha_2 A_3\beta_1\beta_2\beta_3 = \beta_1 I\cdot A_2\cdot\alpha_1\alpha_2 A_2\cdot\beta_2\beta_3 - \beta_2 I\cdot A_2\cdot\alpha_1\alpha_2 A_2\cdot\beta_1\beta_3 + \beta_3 I\cdot A_2\cdot\alpha_1\alpha_2 A_2\cdot\beta_1\beta_2$.)

Finally, the Hamilton product of $\alpha_1, \cdots, \alpha_m$ is $H\cdot\alpha_1\cdots\alpha_m = A_m\cdot\alpha_1\cdots\alpha_m + A_{m-2}\cdot\alpha_1\cdots\alpha_m + \cdots$, understanding that A_{m-m} or $A_0\cdot$ is I. These Hamilton products are associative, that is, for example, $H(H\cdot\alpha_1\alpha_2)(H\cdot\alpha_3\alpha_4\alpha_5) = H\cdot\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = H\cdot\alpha_1 H\alpha_2\alpha_3\alpha_4\alpha_5 =$ etc. The properties of these various expressions are then studied.

24. Mr. Schweitzer pointed out features of his axioms for the projective line analogous to his descriptive axioms for the

line (the system ${}^1R_1$) and for the plane (the system ${}^2R_2$). In particular, he showed how his planar descriptive system could be logically combined with linear projective axioms in such a way that the resulting system might be descriptive or projective, but not necessarily either.

25. In his second paper, Mr. Schweitzer presented a preliminary report on a study of the memoirs of F. Riesz * and H. Hahn † in relation to the generalization of Grassmann's extensive algebra ‡ to a denumerably infinite number of dimensions.

H. E. SLAUGHT,
Secretary of the Section.

THE SIXTY-FIRST MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE sixty-first meeting of the American Association for the Advancement of Science was held in Boston during the convocation week, December 27, 1909, to January 1, 1910. The president of the meeting was President D. S. Jordan, of Stanford University. The address of the retiring president, Professor T. C. Chamberlin, entitled "A geologic forecast of the future opportunities of our race" was given in Sanders Theatre, Cambridge, on the evening of the opening day.

Comparatively few papers on pure mathematics appeared on the separate program of Section A (mathematics and astronomy) because of the fact that the American Mathematical Society held its annual meeting in affiliation with the Association. The address of the retiring vice-president, Professor C. J. Keyser, of Columbia University, entitled "The thesis of modern logistic," was given on Wednesday morning at a joint session of Section A and the American Mathematical Society. At the same session Professor D. E. Smith presented a report on the work of the International Commission on the teaching of mathematics.

Another joint session was held on Tuesday afternoon under the auspices of the mathematicians and the physicists. During

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* *Math. Naturw. Berichte aus Ungarn*, 1905, pp. 309, 341-343, etc.
† *Wiener Berichte*, Abt. IIA, vol. 116, pp. 601, 609-610, 642, etc.
‡ See also *American Journal*, October, 1909, pp. 365-410.