$$f(s) = \int_a^b K(s, t) g(t) dt,$$

where $g(s)$ is any continuous function satisfying

$$\int_a^b p(s) g(s) ds = 0,$$

can be developed into the uniformly convergent series

$$f(s) = \frac{p(s) \int_a^b p(s) f(s) ds}{\int_a^b [p(s)]^2 ds} + \Sigma_i \phi_i(s) \int_a^b \phi_i(s) f(s) ds,$$

where $\phi_i(s)$ are the normalized solutions of (1) and (2).

That this expansion may not hold in case $g(s)$ is any continuous function (as Mr. Cairns states the theorem) is shown by the special example

$$K(s, t) = A(s)p(t) + A(t)p(s) + B(s)B(t),$$

$$A(s) \not\equiv cB(s), \quad \int_a^b A(s)p(s)ds = 0, \quad \int_a^b B(s)p(s)ds = 0,$$

$$g(s) = p(s).$$

THE UNIFICATION OF VECTORIAL NOTATIONS.

Elementi di Calcolo vettoriale con numerose Applicazioni. By C. BURALI-FORTI and R. MARCOLONGO. Bologna, Nicola Zanichelli, 1909. v + 174 pp.

Omografie vettoriali con Applicazioni. By C. BURALI-FORTI and R. MARCOLONGO. Torino, G. B. Petrini, 1909. xi + 115 pp.

1. IN view of the plan that the fourth international congress of mathematicians held at Rome in 1908 should discuss the notations of vector analysis and perhaps lend the weight of its recommendation to some particular system, Burali-Forti and Marcolongo awhile ago set themselves the laudable but somewhat thankless task of collecting and editing all the historical, critical, and scientific material which might be indispensable to a proper settlement of the question by the congress, and this material they published in a series of five notes beginning in

the twenty-third volume (1907) of the *Rendiconti* of Palermo and running through several succeeding numbers and volumes. It is needless to observe that the work was accomplished with the expected accuracy. It was, however, not done with all the completeness desirable. The attention of the authors was turned almost exclusively to the minimum system most useful in mathematical physics, that is, to the questions of addition of vectors, of scalar and vector products, of differentiation with respect to a scalar, and of differentiation with respect to space (gradient of a scalar function and divergence and curl of a vector function of position). Considerable discussion was given to quaternions but the fact was fully recognized by the authors that many subjects which might rightfully have been treated were omitted — among which the linear vector function (Hamilton) or quotients and Lückenausdrücke (Grassmann) or dyadics (Gibbs) were perhaps the most noteworthy.

The authors not only followed their initial program ; they went further and themselves recommended a particular minimum system of notations essentially like any and all of those now employed but differing in the symbols selected. Thus did it appear that these strivers after unification were prone to follow the path of all unifiers and introduce still greater diversity. Seemingly the universal language of vectors, like the universal commercial language, is destined to suffer constantly new amendments at the hands of its zealots. The conception of unification as conceived in the mind of each enthusiastic unifier appears to be that he shall disagree with everybody and that everybody shall then agree with him. For the scalar product of $\mathbf{a}$ and $\mathbf{b}$ the recommendation is $\mathbf{a} \times \mathbf{b}$, for the vector product it is $\mathbf{a} \wedge \mathbf{b}$, for the gradient we have grad, and div and rot for the divergence and curl. The symbol ∇ is abolished. The suggestion $\mathbf{a} \times \mathbf{b}$ for the scalar product seems particularly infelicitous in view of the fact that this notation is in actual use for the vector product. So far as we are aware, this is the first suggestion which violently and confusingly differs from a notation which has become fairly widely established. It is true that Grassmann in some of his papers used the cross as a symbol of scalar multiplication ; but the rare and unimportant historical use of a symbol seems hardly a sufficient reason for the present adoption of the symbol in the face of actual modern usage which is tolerably popular. No such objection can be urged against the use of $\wedge$ for the vector product — that notation is altogether new.

With regard to the necessity for new systems of notations we would point out some facts. First there are already available three elaborately and consistently developed systems which represent a considerable portion of the life-work and life-thought of three great minds, namely, the systems of Hamilton, Grassmann, and Gibbs. Any of these is entirely adequate for usage in physics and in addition each represents a distinct attitude toward the science of multiple algebra. Hamilton's theory was developed from the double point of view of the theory of sets and of the exigencies of an analysis for three dimensional space. It represents algebraically the whole domain of linear associative algebra. Grassmann's work gives a general science of extensive magnitudes in any number of dimensions and contributes to algebra the important idea that a product need not be a quantity of the type of either factor. It is the prime geometric algebra. Gibbs's view seems to have been to fuse all these elements and the theory of matrices into a general science of multiple algebra (as distinguished from many individual sciences of different multiple algebras) and to construct from his general point of view a particular system especially fitted for his fellow physicists. In addition to these great scientific analyses of space we have the analysis adopted by the Encyclopedia and many present writers, especially in Germany. This system may apparently be characterized as opportunist in that it seems to have been selected largely in haphazard fashion as a sort of convenient abridged notation. From the sentimental view of scientific fitness and justice it would appear desirable to adopt, if any definitive adoption must be made, one of the systems connected with the name of a great scientist and constructed on scientific principles. From the practical point of view it might be convenient to adopt the opportunist system already so widely used. The hue and cry about the confusion due to the great diversity of notation is largely hysterical. An examination of current literature will show that there is very little diversity and that of the works currently written in vectorial notation not only a plurality but an actual and considerable majority are written in the opportunist system. The chief reason vectors are not more used is not this alleged confusion and diversity so much as it is inertia. The "news items" and "information" and "reasons" which are spread broadcast concerning vectors are about as true, about as fundamental, and about as much founded in fact as those scattered into the air about the markets for stocks and commodities. With regard to new systems we may mention that Gibbs, with a modesty not universally in evidence, for many years refrained from publishing his system and from allowing it to be published (although he privately printed it for his own convenience and for that of his students) because he felt it did not present sufficiently original differences from existing methods. His attitude was perhaps unfortunate. The world may have been in need of the new systems; it may still be in need of new systems; this, at any rate, is the opinion of the authors whose works are under review.

Further to spread the new system and to offer to Italian scientists and students a treatise on vector analysis in their own language, the authors have written two short volumes of which the titles appear at the head of this review. It is surely a good thing that there is now at least one systematic treatment of vector analysis in Italian. The first volume, Calcolo vettoriale, takes up the matters mentioned as belonging to the minimum system; the second, Omografie vettoriali, discusses what is equivalent to the linear vector function. In their prefaces the authors state that their books differ profoundly both in method and in notation from all previous texts of recent years — differ in method because they intend to operate in an absolute manner on geometric entitiés, whereas ordinarily vectors and their operations are tachygraphic for coordinates; and differ in notations by the adoption of their own recommendations. It will be impossible for us to discuss these books apart from the contributions of the authors to the *Rendiconti* and to *L'Enseignement Mathématique*, which has so considerately opened its columns to an interchange of views by all interested in vectorial notations. Quotations will be made indifferently from all these sources according as they may be needed.

2. In commencing the detailed review of the two books, mention should be made of the fact that they contain not so much a pure vector analysis as a point and vector analysis with the emphasis on vectors. Thus the vector is introduced as the difference $B - A$ of two points and such equations as

$$P = A + x(B - A), \quad P = O + x\mathrm{i} + y\mathrm{j} + z\mathbf{k},$$

for a line and for the position of a point are found. There is much to be said for this procedure, which introduces the origin explicitly into the analysis in addition to the system **i, j, k.** Its advantages and disadvantages reative to the usual pure vector analysis will be clear to all readers. On the mathematical and logical side this usage of the authors is highly commendable; it opens the way for the consideration of Grassmann's geometrical algebra and eliminates the origin wherever unessential. Whether physicists, who have a tendency to limit their analysis to an irreducible minimum, will welcome the addition of a little point analysis is doubtful. In other respects the first three chapters give a treatment of addition and multiplication of vectors in much the usual way and with much the usual applications.

With Chapter IV there is an interesting departure from the ordinary texts. The operator i, as a quadrantal versor in a plane, is introduced by the equation * $i\mathbf{x} = \mathbf{u} \times \mathbf{x}$, where $\mathbf{x}$ is normal to $\mathbf{u}$. That this operator i, as regards its repetition, obeys the law $i^2 = -1$ of $i = \sqrt{-1}$ is pointed out. The equations

$$(i\mathbf{a}) \cdot (i\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}, \quad \mathbf{a} \times (i\mathbf{b}) = \mathbf{b} \times (i\mathbf{a}),$$

however, show the reader at once that here the i is not an ordinary scalar subject to $i^2 = -1$. For if this were the case, $i\mathbf{a} \cdot i\mathbf{b}$ would be $-\mathbf{a} \cdot \mathbf{b}$, and like changes of sign would occur in other formulas. This fact taken with the fact that the operator i really depends on the vector $\mathbf{u}$ which enters into its definition seems to militate greatly against the usefulness of this chapter and the advisability of its general acceptance. It appears unfortunate to use a symbol so familiar as i in a sense which precludes its fuller treatment according to the same formal laws as it ordinarily obeys. This impression is but strengthened by the sight of the equations

$$e^{i\phi}\mathbf{a} \cdot e^{i\phi}\mathbf{b} = \mathbf{a} \cdot \mathbf{b}, \quad e^{i\phi}\mathbf{a} \cdot e^{i\psi}\mathbf{a} = \mathbf{a}^2 \cos(\psi - \phi),$$

and others of that ilk for rotations through various angles. It may well be, however, that the unfavorable impression is due merely to the unfamiliarity of the symbols, or rather, to a familiarity with them under the form of scalars subject to the laws of (complex) scalars.

After a brief mention, in Chapter V, of the differentiation of a vector with respect to a scalar the vitally important subject of differentiation with respect to space, that is, of gradient,

* Here and throughout the review the notations of the authors are translated so far as possible into the notations of Gibbs, which are probably more familiar to readers of the BULLETIN.

divergence, and curl, is treated in Chapter VI. The gradient is defined by the equation *

$$(1) \qquad dV = (\text{grad } V) \cdot dP.$$

Two remarks may be made. In the first place this definition is intrinsic or absolute, that is, devoid of reference to coordinate axes, and therein conforms to the usage we have always indicated as the best for defining ∇V. The second observation is that the authors here and elsewhere use dP, the differential of a point, instead of $d\mathbf{r}$, the differential of a vector. This has the advantage that no origin is implicated in the definition. Whether this advantage is sufficient to compensate for the slight complication of the combined point and vector analysis as against the simple vector analysis is a matter of individual opinion. The authors have a violent dislike for the symbol ∇ and not only give up its use but urge vehemently that it should be universally abandoned. This matter will be discussed later in section 4.

The defining equations for curl $\mathbf{V}$ and div $\mathbf{V}$ are given in the absolute vectorial form

$$(2) \qquad d\mathbf{r} \times \delta\mathbf{r} \cdot \text{curl } \mathbf{V} = \delta\mathbf{r} \cdot d\mathbf{V} - d\mathbf{r} \cdot \delta\mathbf{V}.$$

$$(3) \qquad \text{div } \mathbf{V} = \mathbf{a} \cdot [\text{grad } (\mathbf{a} \cdot \mathbf{V}) + \text{curl } (\mathbf{a} \times \mathbf{V})],$$

where $d\mathbf{r}$, $\delta\mathbf{r}$ are two differential displacements (written dP, δP in the text) and $d\mathbf{V}$, $\delta\mathbf{V}$ are the corresponding differentials of $\mathbf{V}$; and where the equation (3) is supposed to hold for any constant vector $\mathbf{a}$; and where finally we have seen fit for ulterior purposes to rearrange the order of the vectors in the products. These are highly ingenious definitions and have much to commend them. It will be noticed that (2) is suggested at once by the expressions that occur in the proof of Stokes's theorem by the method of variations; the definition of the divergence appears more artificial and is less intimately connected with the fundamental characteristics of div $\mathbf{V}$; both, however, are well adapted to establish some of the important formulas of the differential calculus of vectors and should therefore have the serious attention of students of the presentation of vector analysis.

It should be noticed that as defined by (1) the gradient is immediately interpretable in its physical sense as that vector

* Compare, in Gibbs's notation, $d\mathbf{r} \cdot \nabla V = dV$.

which gives in direction and magnitude the most rapid rise of V. Now the basal significance of curl $\mathbf{V}$ is found in the fact expressed by Stokes's theorem that the induction of curl $\mathbf{V}$ through a surface is equal to the integral of $\mathbf{V}$ around the boundary, and the basal significance of div $\mathbf{V}$ lies in Gauss's theorem that the integral of div $\mathbf{V}$ over a volume is equal to the induction of $\mathbf{V}$ through the bounding surface. Hence for physical reasons definitions like

$$(2') \qquad d\mathbf{S}\cdot\operatorname{curl}\mathbf{V} = \int_{\bigcirc} d\mathbf{r}\cdot\mathbf{V},$$

$$(3') \qquad d\tau \operatorname{div} \mathbf{V} = \int_{\bigcirc} d\mathbf{S}\cdot\mathbf{V},$$

where $d\mathbf{S}$ is an element of surface and $d\tau$ an element of volume, are preferable to the analytic definitions of the authors, and they have the added felicity to be immediately connected with the proofs of Stokes's and Gauss's theorems. They have the disadvantages that they are not so easily available for the proof of formulas of differential calculus and that they mix differential and integral calculus. Whether on the whole the advantages are with (2′), (3′) as against (2), (3) must remain largely a matter of personal preference.

3. It is in this same chapter that the authors reveal their remarkable discovery that the laplacian operator

$$\triangle = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is essentially different according as it is applied to a scalar or to a vector function. Upon this discovery they are especially insistent on every possible occasion.* They even go so far as to introduce different symbols $\triangle$ and $\triangle'$ for the operator according as the operand is scalar or vector. Then they are able to write

$$(4) \qquad \begin{aligned} \triangle V &= \operatorname{div} \operatorname{grad} V, \\ \triangle'\mathbf{V} &= \operatorname{grad} \operatorname{div} \mathbf{V} - \operatorname{curl} \operatorname{curl} \mathbf{V}. \end{aligned}$$

Whereupon they add: You see at once what an immense difference there is between the two operators $\triangle$ and $\triangle'$ for which

* Calcolo vettoriale, p. 72; Omografie vettoriali, p. 61; and *L'Enseignement Mathématique* (1909), p. 462, from which the quotation below is taken.

authors have adopted the single symbol $\triangle_2$ because these operators have the same cartesian expression. That is no reason to use a single sign to designate very different things; on the contrary it shows once more that the systematic use of coordinates may introduce pseudo-operators which have no longer a geometric and logical character.

Now it is rather curious and decidedly regrettable that two such eminent authors, who work in a country conspicuous for its researches in the logic of mathematics and of whom one is himself illustrious for such investigations, should adduce the equations (4) as a reason for the essential difference of $\triangle$ and $\triangle'$. It merely goes to show how restricted is their point of view in the mathematics of the subject and how deficient in the physical interpretation thereof. If the authors were not so eminent and if we did not have so high a regard for most of their work in mathematics and mathematical physics, we should pass over these remarks of theirs without further comment and in full confidence that no very great number of mathematicians or physicists would fall into their way of thinking. Unfortunately that course is not open to us. The higher the sources from which vicious doctrines are promulgated, the more patient and painstaking must be their refutation. All that equations (4) do is to give two different expressions which are equivalent in the two respective cases to the laplacian operator. The second expression cannot be applied in the first case because the operators div and curl cannot be applied to a scalar; the first expression can be applied in the second case if one knows how to define the gradient of a vector function and the divergence of a linear vector function, but the authors do not give these definitions and there is no need to give them now.*

The laplacian operator is probably best known to mathematicians in connection with Laplace's equation and harmonic functions. It is a fundamental theorem on harmonic functions that the average value of the function upon the surface of a sphere is equal to the value at the center, and it is an immediate corollary that the average value throughout the sphere is equal to the value at the center. Now as the laplacian operator occurs so frequently in fundamental physical problems, it is a reasonable assumption that the operator represents some intrinsic or absolute characteristic of a field and does not depend in any other than an accidental way upon cartesian coordinates.

*See, however, (1), (2), (3) of section 7 below.

And Maxwell pointed out this characteristic which may be called the dispersion. The laplacian operator may in fact be defined by either of the equivalent intrinsic or absolute equations

$$(5) \qquad r^2 \triangle \phi = 6(\bar{\phi} - \phi) = 10(\bar{\bar{\phi}} - \phi),$$

where r is the radius of an infinitesimal sphere and $\bar{\phi}$, $\bar{\bar{\phi}}$ are respectively the average value of ϕ upon the surface and throughout the volume of the sphere. This definition is comparable to (2′), (3′) for divergence and curl in that it depends on integration. But it is incomparably simpler than either of them. For they depend upon scalar products, whereas this depends only on addition as implied in averaging.

In other words the laplacian operator may and should be defined by its intrinsic properties such as are expressed in (5), and when this definition is given it appears that the operator is equally applicable and with the same significance to any quantity ϕ which satisfies the laws of addition, that is, to the elements ϕ of any linear algebra regarded as functions of position and in particular to scalar functions, vector functions, dyadic or linear-vector-function functions, and to planar vector or bivector functions. The laplacian operator is in no wise a pseudo-operator, there is no immense nor even any slight essential difference in its application to various linear fields, and there is no more reason for representing it by different symbols than there is for representing addition by different symbols. With all these facts Maxwell was familiar and so was Gibbs. The really accidental phenomenon is that the laplacian operator can be expanded in either of the forms given by (4).

4. It has been remarked that the authors do not use ∇ for grad, and it is a corollary that they should not use $\nabla \times$ and $\nabla \cdot$ for curl and divergence, which they write as rot and div. In regard to all these operators they remark that: The operators $\nabla \cdot$, $\nabla \times$ are absolute because div and rot may be defined without coordinates as in (2), (3); but ∇, on the contrary, is an essential tachygraph and the sign $\times$ or $\cdot$ which follows it is an operator which without coordinates has no meaning.* It is apparently for such reasons as these that they discard ∇ from their calculus. Now it fails to appear clear to us why grad V or ∇V is not defined by (1) quite as absolutely and just as independently of coordinates as curl $\mathbf{V}$ or $\nabla \times \mathbf{V}$ by (2) and div $\mathbf{V}$ or $\nabla \cdot \mathbf{V}$ by (3). Inasmuch as the definitions ∇V, $\nabla \cdot \mathbf{V}$,

* *L'Enseignement Mathématique* (1909), p. 466.

$\nabla \times \mathbf{V}$ are identical with those for grad V, div $\mathbf{V}$, curl $\mathbf{V}$, or may be taken so, any dependence of one set on coordinates should establish an equal dependence for the others — and we believe that in neither case is there any such dependence.

The question, however, does arise as to the availability of the detached symbols ∇, $\nabla\cdot$, $\nabla\times$, or grad, div, curl. Now in case three totally different symbols like the last set are chosen for the three types of differentiation, there can be little object in separating the operators from the operands and no analytic algorism is suggested. But if the symbols ∇, $\nabla\cdot$, $\nabla\times$ are used and if their use indicates a simple and suggestive algorism by means of which differential formulas may be remembered, then the detaching of the operators from the operand or better the introduction of the idea that the symbols ∇, $\nabla\times$, $\nabla\cdot$ should be interpreted as a combination of the operations ∇, $\times$, $\cdot$ is forcibly recommended.* For those who scorn analytic algorisms and prefer to remember a lot of distinct formulas, this argument is not impressive; but by far the larger number of persons like to have a notation which is algorismic in the sense that in itself it suggests the proper analytic transformations, and there is a very wide belief that there is much of mathematical value in a notation which has the felicity to be suggestive. The question, then, as to the preference of ∇, $\nabla\cdot$, $\nabla\times$ over grad, div, curl, is not one of dependence or independence of coordinates but one of analytic felicity.

To investigate this matter more closely, let it be granted that ∇, $\nabla\cdot$, $\nabla\times$ represent some sort of differentiation so that they may appropriately be called differentiating operators and may be expected to obey the fundamental laws of differentiation

$$(6) \qquad D(u+v) = Du + Dv, \quad D(uv) = D_u(uv) + D_v(uv),$$

where the subscripts u and v denote that u and v respectively are considered as differentiated subject to the constancy of the other. As ∇V is a vector when V is scalar, the operator ∇ may properly be called a vector operator and hence a vector differentiating operator. Next it is but natural to observe that the vector

* These remarks and those which follow, although written in the notation of Gibbs, should not be interpreted as limited to any particular system of notation; the recommendation is merely that, whatever be the notation for the scalar and vector products, the notation should be preserved and combined with ∇ or some equivalent sign of differentiation to express the divergence and curl of a vector function, while ∇ or its equivalent gives the gradient of a scalar function.

characteristic of ∇ is maintained in $\nabla \cdot \mathbf{V}$ and $\nabla \times \mathbf{V}$ provided that the $\cdot$ and $\times$ be considered in their usual general significance as operators which respectively combine two vectors into a scalar and into a vector. That there may be less chance for misconceiving our meaning, it may be stated that up to this time none of the operators is supposed to have had any cartesian interpretation and that the $\cdot$ and $\times$ cannot be considered as representing scalar and vector products of ∇ by V in the usual sense. The entire aim is to examine the formulas for differentiation with a view to determining whether or not the symbols obey the laws (6) of differentiation and the laws of scalar and vector products to an extent sufficient to warrant the statement that they are analytically suggestive.*

The formulas of differentiation as given by the authors are

$$\nabla(cV) = c\nabla V, \quad \nabla \cdot \mathbf{C} = 0, \quad \nabla \times \mathbf{C} = 0,$$
$$\nabla(U + V) = \nabla U + \nabla V, \quad \nabla \cdot (\mathbf{U} + \mathbf{V}) = \nabla \cdot \mathbf{U} + \nabla \cdot \mathbf{V},$$
$$\nabla \times (\mathbf{U} + \mathbf{V}) = \nabla \times \mathbf{U} + \nabla \times \mathbf{V}, \quad f(V)\nabla V = \nabla \int f(V)\,dV,$$
$$\nabla \times (V\mathbf{U}) = V\nabla \times \mathbf{U} + (\nabla V) \times \mathbf{U},$$
$$\nabla \cdot (V\mathbf{U}) = V\nabla \cdot \mathbf{U} + (\nabla V) \cdot \mathbf{U},$$
$$\nabla \cdot (\mathbf{V} \times \mathbf{U}) = \mathbf{U} \cdot \nabla \times \mathbf{V} - \mathbf{V} \cdot \nabla \times \mathbf{U}, \quad \nabla \times (\mathbf{V} \times \mathbf{U}) = ??$$

The results of the first three lines are certainly suggested by the fact that ∇ is a differentiating operator and subject to the first of equations (6). The results of the next line are similarly suggested by the second of equations (6). To show the procedure in greater detail:

$$\begin{aligned}\nabla \times (V\mathbf{U}) &= \nabla_V \times (V\mathbf{U}) + \nabla_{\mathbf{U}} \times (V\mathbf{U}) \\ &= \nabla_V V \times \mathbf{U} + V\nabla_{\mathbf{U}} \times \mathbf{U}.\end{aligned}$$

The scalar V when variable is passed out next to ∇ for differentiation, leaving the $\times$ between two vectors ∇V, $\mathbf{U}$; and the scalar V when constant is passed out past the sign of differentiation. The formula of the last line is treated in a similar way, but the additional laws for the interchange of dot and cross in

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* The fact that ∇ may be expanded as $\mathbf{i}(\partial/\partial x) + \mathbf{j}(\partial/\partial y) + \mathbf{k}(\partial/\partial z)$ and that then curl $\mathbf{V}$ and div $\mathbf{V}$ may be regarded as the formal products $\nabla \times \mathbf{V}$ and $\nabla \cdot \mathbf{V}$ obtained according to the laws of multiplication would, from the present point of view, be the last instead of the first argument in justification of the notations.

a triple product or for the interchange of order in a vector product are used.

$$\begin{aligned}\nabla\cdot(\mathbf{V}\times\mathbf{U}) &= \nabla_{\mathrm{V}}\cdot(\mathbf{V}\times\mathbf{U}) + \nabla_{\mathrm{U}}\cdot(\mathbf{V}\times\mathbf{U})\\ &= \nabla_{\mathrm{V}}\times\mathbf{V}\cdot\mathbf{U} - \nabla_{\mathrm{U}}\cdot(\mathbf{U}\times\mathbf{V})\\ &= (\nabla\times V)\cdot\mathbf{U}-(\nabla\times\mathbf{U})\cdot\mathbf{V} = \mathbf{U}\cdot\nabla\times\mathbf{V}-\mathbf{V}\cdot\nabla\times\mathbf{U}.\end{aligned}$$

The fact that the notation suggests the result is clear.

It is desirable to go somewhat further. The authors remark * that $\nabla\times(\mathbf{V}\times\mathbf{U})$ cannot be expressed in terms of grad, div, and curl, and they use libellous language toward the operator $\mathbf{a}\cdot\nabla$ — we believe they imply that it is an essential tachygraph! Now we have a fondness for this operator because, if $\mathbf{a}$ is a unit vector, it gives the directional derivative of a vector function.† We propose therefore to inject this operator into their system in a manner quite in sympathy with their definitions of curl and divergence. Consider

$$(7)\quad \mathbf{a}\times(\nabla\times\mathbf{V}) = (\mathbf{a}\cdot\mathbf{V})\,\nabla - (\mathbf{a}\cdot\nabla)\,\mathbf{V} = \nabla\,(\mathbf{a}\cdot\mathbf{V}) - (\mathbf{a}\cdot\nabla)\,\mathbf{V},$$

where $\mathbf{a}$ is a constant vector. This formula has been obtained by expanding $\mathbf{a}\times(\nabla\times\mathbf{V})$ just as $\mathbf{a}\times(\mathbf{b}\times\mathbf{V})$ would be expanded and by the transference of ∇ in the first term to a position where it differentiates $\mathbf{V}$ as it should. Whether or not this procedure is regarded as accurate, the definition may be given that

$$(7')\qquad (\mathbf{a}\cdot\nabla)\mathbf{V} = \nabla\,(\mathbf{a}\cdot\mathbf{V}) - \mathbf{a}\times(\nabla\times\mathbf{V}),$$

and this definition of $(\mathbf{a}\cdot\nabla)\,\mathbf{V}$ is no more artificial than their definition (3) of div $\mathbf{V}$. Next let $d\mathbf{r}$ be any vector, consider $\mathbf{a}$ as the differential $\delta\mathbf{r}$, and apply the definition (2) of curl.

$$d\mathbf{r}\cdot\mathbf{a}\times(\nabla\times\mathbf{V}) = \mathbf{a}\cdot d\mathbf{V} - d\mathbf{r}\cdot\delta\mathbf{V}, \qquad \text{by (2)},$$

$$d\mathbf{r}\cdot\nabla(\mathbf{a}\cdot\mathbf{V}) = d(\mathbf{a}\cdot\mathbf{V}) = \mathbf{a}\cdot d\mathbf{V}, \qquad \text{by (1)}.$$

$$\therefore d\mathbf{r}\cdot\left[(\mathbf{a}\cdot\nabla)\,\mathbf{V}\right] = \mathbf{a}\cdot d\mathbf{V} - \mathbf{a}\cdot d\mathbf{V} - d\mathbf{r}\cdot\delta\mathbf{V} = d\mathbf{r}\cdot\delta\mathbf{V}.$$

$$(8)\qquad \therefore (\mathbf{a}\cdot\nabla)\,\mathbf{V} = \delta\mathbf{V} \quad\text{or}\quad (\delta\mathbf{r}\cdot\nabla)\,\mathbf{V} = \delta\mathbf{V}.$$

The last result is found by canceling the arbitrary vector $d\mathbf{r}$.

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* Calcolo vettoriale, p. 68, Omografie vettoriali, p. 51.

† The directional derivative, like the laplacian operator, is a scalar operator and, like it, may be applied to the elements ϕ of any linear field when ϕ is regarded as a function of position. The definition as given in (7′) does not, however, apply in the general case, owing to the vector operations which it contains. In this respect (7′) corresponds to (4) and not to (5).

It is seen from (8) that $(d\mathbf{r} \cdot \nabla)\mathbf{V} = d\mathbf{V}$ and hence that if $\mathbf{a}$ is a unit vector $(\mathbf{a} \cdot \nabla)\mathbf{V}$ is the directional derivative of $\mathbf{V}$ in the direction $\mathbf{a}$. This result may be compared with (1). When V is scalar, ∇V is defined and $d\mathbf{r} \cdot \nabla V = dV$; the operation $(d\mathbf{r} \cdot \nabla) V$ is not defined in this case, but may naturally be defined as equal to $d\mathbf{r} \cdot \nabla V$. When $\mathbf{V}$ is a vector, $\nabla\mathbf{V}$ is not defined but $(d\mathbf{r} \cdot \nabla)\mathbf{V}$ is defined and the probability is suggested that at some future time it may become convenient to define $\nabla\mathbf{V}$ by the relation $d\mathbf{r} \cdot \nabla\mathbf{V} = (d\mathbf{r} \cdot \nabla)\mathbf{V} = d\mathbf{V}$. At any rate with this addition of the directional derivative to the system, expressions for $\nabla \times (\mathbf{V} \times \mathbf{U})$ and $\nabla (\mathbf{V} \cdot \mathbf{U})$* may be found and are identical with those suggested by the formal method.

A word may be added concerning derivatives of the second order. Here there are three important identities.

$$\nabla \cdot \nabla \times \mathbf{V} = 0, \quad \nabla \times \nabla V = 0,$$
$$\nabla \times (\nabla \times \mathbf{V}) = \nabla\nabla \cdot \mathbf{V} - \nabla \cdot \nabla\mathbf{V}.$$

Of these the first two are naturally suggested by the vector characteristics of ∇ and the third, which is equivalent to (4), tallies with the formal expansion analogous to that used in (7). Now it is by no means our intention to regard these formal derivations of the fundamental formulas of differentiation as proofs of those formulas, but merely as proofs of the statement that the notations involving ∇ are analytically suggestive and that for this reason these notations are far preferable to those like grad, div, curl. It may perfectly well be that they have disadvantages which should cause their abandonment; but these disadvantages should be clearly stated and the statements should be true and not mistaken. The only statement which we have seen and which we regard as true relative to the disadvantages of the notations ∇, $\nabla \cdot$, $\nabla \times$ is that they do not suggest the physical significance of the operations as well as grad, div, curl do. Whether this nominal infelicity outweighs the analytic suggestiveness must be left to individual opinion.†

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* The result $\nabla(\mathbf{U} \cdot \mathbf{V}) = (\nabla\mathbf{U}) \cdot \mathbf{V} + (\nabla\mathbf{V}) \cdot \mathbf{U}$ which the formal method indicates is correct; but, as it is meaningless without definitions of $\nabla\mathbf{U}$ and $\nabla\mathbf{V}$, another form involving the directional derivative and the curl is usually given when treating only the minimum system.

† The reader will recall that at times there have been serious objections urged against replacing the S and V in the notations $S\mathbf{ab}$ and $V\mathbf{ab}$ for the scalar and vector products by other symbols, for the reason that no other notation so forcibly suggests which product is scalar and which is vector; nevertheless the S and V are not used much now.

5. The first half of the Calcolo vettoriale has now been covered. The second half, which is entitled applications, starts with a treatment of geometric applications, especially the tangent, normal, and binormal to curves. Here the use of vectors has decided advantages over the usual methods.

The second chapter on applications deals with fundamental theorems of integral calculus more than with applications. The formulas

$$\int \nabla \times \mathbf{V} d\tau = \int d\mathbf{S} \times \mathbf{V}, \quad \int \nabla V d\tau = \int V d\mathbf{S}, \quad \int \nabla \cdot \mathbf{V} d\tau = \int \mathbf{V} \cdot d\mathbf{S},$$

which are equivalent to an integration are given and proved. These formulas and many similar ones are all consequences of the operational equation (where any sign or no sign of multiplication may be inserted after ∇ and $d\mathbf{S}$)

$$\iiint d\tau \nabla(\ ) = \iint d\mathbf{S}(\ )$$

of which the proof or for which a justification may easily be given. Green's theorem, Stokes's theorem, and the formula for the rate of change of the flux of a fluid through a surface are the other topics of the chapter. The applications to mechanics come next and are tolerably numerous — velocity and acceleration of a point and their resolutions along various directions, central motion, kinematics of a rigid body, motion of a rigid body in its plane, equilibrium of strings, motion of a rigid body in space. The selection of topics and the method of discussion are both admirable. It is here that the authors find considerable use for their operator $e^{i\phi}$ which establishes a rotation through the angle ϕ.

The applications of a more physical character follow. A short chapter on hydromechanics sets forth the derivation of the equations of motion, vortical motion, and velocity potential. It is interesting to remark how short, direct, and elegant is the treatment of these fundamental questions by vectorial methods. The elements of the theory of the equilibrium of an isotropic elastic body is given. The volume closes with applications to electromagnetism including retarded potentials, Maxwell's equations, the Poynting vector, integrals of the equations, and finally Lorentz's equations. It may therefore be seen that this volume affords a very good introduction to the elementary general theories of mathematical physics in addition to its presentation of the most important parts of vector analysis. The work cannot fail to be of interest to physicists and to students of vectorial methods. In it may be found numerous suggestions that merit wide-spread adoption. The large amount of material which has been put into a small space without any apparent crowding or obscurity is especially noteworthy and has been accomplished largely by adherence to the program of using purely vectorial methods. Somewhat greater reference to coordinates might have made the work easier to read for those who previously were unacquainted with vectors, but certain compensating disadvantages would undoubtedly have arisen.

6. The second volume, Omografie vettoriali, is divided into three chapters which treat respectively homographies or linear transformations with a fixed origin, differentiation with respect to a point, and applications to mathematical physics. In addition there is an introduction which presents a few generalities on linear operators or transformations and an appendix which contains some added developments in the analytic theory of homographies and some applications to the differential geometry of surfaces. The linear transformation or homography is designated by a small Greek letter and the vector which results from the application of the homography α to a vector $\mathbf{u}$ is denoted by $\alpha\mathbf{u}$. The invariants of α are written as $I_3\alpha$, $I_2\alpha$, $I_1\alpha$ and are defined by the equations

$$I_3\alpha = \frac{\alpha\mathbf{u} \times \alpha\mathbf{v} \cdot \alpha\mathbf{w}}{\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}}, \quad I_2\alpha = \left[\frac{d}{dx} I_3(x + \alpha)\right]_{x=0}$$

with an additional equation involving the second derivative to define $I_1\alpha$. It is shown that $I_3\alpha$ is independent of the three independent vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ which enter into its definition and that consequently the other two invariants are also really invariants. The reader will observe that the authors have no hesitation about adding together a number x and a homography α. They regard a number, whenever convenient, as a linear transformation. The meaning of the expression $x + \alpha$ would probably be taken from the equation

$$(x + \alpha)\mathbf{u} = x\mathbf{u} + \alpha\mathbf{u}, \quad \mathbf{u} \text{ arbitrary.}$$

Now although this equation may be regarded as a justification of the usage of the authors, the question does arise as to whether or not their usage is really good. From the algebraic or matricular points of view, which they almost wholly ignore, the quantity α is an element of a quadrate algebra containing nine units and having a modulus (often called the idemfactor), but the unit 1 is not one of the units as it is in the case of the ordinary complex numbers or in the case of quaternions, and the idemfactor, which in those cases is 1, is not 1 in the algebra of homographies α because 1 is not an element of the system. For reasons that perhaps are merely puristic we believe that it is better to regard 1 and the idemfactor as distinct and to refrain from identifying multiplication by a number with multiplication in the system. Statements like : If α, β are homographies and m is a real number, then

$$I_1(\alpha + \beta) = I_1\alpha + I_1\beta, \quad I_1(m\alpha) = mI_1\alpha, \quad I_1 m = 3m,$$

seem particularly confused and infelicitous.

After a brief mention of singular homographies, of the fixed directions of a homography, and of the identical equation which a homography satisfies, the authors pass to the consideration of various types of homographies. According as one of the equations *

$$\mathbf{x} \cdot \alpha\mathbf{y} = \mathbf{y} \cdot \alpha\mathbf{x} \quad \text{or} \quad \mathbf{x} \cdot \alpha\mathbf{y} + \mathbf{y} \cdot \alpha\mathbf{x} = 0$$

may be satisfied, the homography is called a dilatation or an axial homography. These correspond to what are often called self-conjugate and anti-self-conjugate or symmetric and skew-symmetric matrices or linear vector functions. The symbols D and V are introduced so that $D\alpha$ shall represent the dilatation of α or self-conjugate part of α and $V\alpha$ shall represent the vector of the axial remainder $\alpha - D\alpha$ of α. The result is that α may be written as

$$\alpha = D\alpha + V\alpha\times, \quad \text{and} \quad K\alpha = D\alpha - V\alpha\times$$

is thereupon taken as the definition of the conjugate of α. Both these equations must be regarded as operational equations and not as equations in multiple algebra. In this respect they are like $x + \alpha$. To obtain the transformation of plane areas regarded as vectors, the symbol R is introduced by the definition

$$R\alpha(\mathbf{x} \times \mathbf{y}) = \alpha\mathbf{x} \times \alpha\mathbf{y}, \quad \text{and} \quad C\alpha = I_1\alpha - \alpha$$

is another definition which is reserved, however, for the appendix. There follow a large number of formulas connecting

* Readers familiar with current notations in linear associative algebra or with the notations of Hamilton, Gibbs, Cayley, or Clebsch-Aronhold for such equations as these will notice the lack of symmetry in the authors' use of the sign of multiplication and probably will regret that the idea of a post-operator was not introduced on a par with that of a pre-operator ; this, however, would have involved the authors in serious notational difficulties.

the symbols I_r, D, V, K, R as applied to homographies (including real numbers). An additional symbol H is introduced so that $H(\mathbf{u}, \mathbf{v})$ may represent what in Gibbs's system is the dyad **vu**. More relations between the symbols are given. Finally the chapter ends with a discussion of singular homographies and of versors and perversors.

From what has been said it will appear that the point of view of the authors is operational and not algebraic. Whether by taking this method and introducing these symbols they have materially added to the unity or unification of vectorial notations may be debated. Had they not been so insistent on the necessity of unification, we should say that these new methods and notations added an interesting and instructive diversity to the subject, and that the more points of view we had the better off we were both in a scientific and in a pedagogical way. Undoubtedly the best and most thorough way for anybody to learn a subject that is new and unfamiliar and unsatisfactory to him is to rewrite the subject according to his own desires; this replaces mere receptivity by original activity and lends a zest to the study. We should be happy to see everyone who is interested in vectors and who believes in their necessity adopt the authors' method and make the analysis suit himself. The adherents of unification would probably regard this proffered liberty as an invitation to license in any case other than their own. In fact although they are perfectly willing to use the operators i and $e^{i\phi}$ in a somewhat unusual way and to add numbers and homographies in a manner not in accord with the most careful practice, they are unwilling to let others use a single symbol for the laplacian operator or the notations ∇ and $\mathbf{a} \cdot \nabla$ which are of long standing and like the laplacian operator are almost universally believed to be essential operators independent of the axes of reference.

The question of the choice between the operational and algebraic treatment of strains deserves the most careful consideration if the proposition to abandon one and confine the attention to the other is seriously maintained. Linear operators are important. Owing to the researches of Volterra, Hadamard, Fréchet, and others, their importance is becoming rapidly extended to the domain of higher analysis. Hence it seems as though so much of the theory of linear operators as is applicable in general should be presented in the treatment of strains. The real question is whether the method should be applied to strains in a detail which is not available in general and to the exclusion of the algebraic treatment. Without going into the matter of multiple algebra in general, mention may be made of the very important subject of matrices, which may now be studied so readily in the excellent presentation given by Bôcher in his Introduction to higher algebra. That there is a sort of isomorphism between matrices of the third order and strains can hardly be denied, and the isomorphism extends to matrices of higher order and strains in spaces of higher dimensions. Apart from the introduction of an arbitrary factor of proportionality, collineations in $n-1$ dimensions may replace strains in n dimensions. The subject of matrices is naturally so presented that one who is familiar with it cannot be said to be unfamiliar with the theory of strains, especially as that theory is presented by Gibbs. And it seems only fair toward the student who for any reason learns the theory of strains to present that theory in a way which makes it conversely true that one who is familiar with the theory of strains cannot be said to be really unfamiliar with the theory of matrices. It is doubtful if the operational method is fair in this sense. Moreover although the algebraic method may not always appear quite so direct as the operational, it has the advantage of possessing an important algorism — the algorism of multiplication — which adds considerably to the ease of acquisition and offers some insight into the general domain of multiple algebra. In fine, the algebraic method seems to us to afford so much of the operational point of view as is useful for the general theory of linear operators and in addition to offer intimate points of contact with the theory of matrices and the theory of multiple algebra.*

7. The chapter on derivatives is shorter but none the less important. If u is any entity which is a function of position, the authors define the derivative

$$\frac{du}{dP} \quad \text{by} \quad \frac{du}{dP}dP = du, \tag{9}$$

and they point out that the derivative of a vector is a homo-

* We believe that a large number of writers fail to bring out clearly and perhaps even fail to realize the setting of the simple vector algebra in the wide domain of general multiple algebra. To this is attributable many a comment founded exclusively on a too restricted point of view. Whatever may be advisable pedagogically, the only scientific aspect of vector analysis is in its place in multiple algebra.

graphy and the derivative of a homography is a linear operator which converts a vector into a homography. If $\mathbf{u}$ is a vector, the derivative of $\mathbf{u}$ is not the linear vector function $\nabla \mathbf{u}$ of Gibbs; it is the conjugate $(\nabla \mathbf{u})_C$ of that function, because the authors apply the differential vector dP after the operator where Gibbs writes $d\mathbf{r} \cdot \nabla \mathbf{u} = d\mathbf{u}$ and applies it before. In like manner the derivative of a homography is not $\nabla \alpha$, but one of its five conjugates. It may be noted that, as the derivative $d\mathbf{u}/dP$ of the present authors and the derivative $\nabla \mathbf{u}$ of Gibbs are both linear vector functions, they differ from Hamilton's $\nabla \mathbf{u}$ which is a quaternion. Numerous formulas for differentiation are given. It should be remarked that Gibbs was familiar with the authors' notation for derivatives but abandoned it in favor of the notation ∇u, for reasons known probably to no one. Victor Fischer in his Vektordifferentiation und Vektorintegration resumes this notation and extends it by using a dot and cross in a manner suggested by $\nabla \cdot \mathbf{u}$ and $\nabla \times \mathbf{u}$.

The authors next proceed to define a vector grad α by

$$(10) \qquad \operatorname{grad} \alpha = \left(\frac{d\alpha}{dP}\mathbf{i}\right)\mathbf{i} + \left(\frac{d\alpha}{dP}\mathbf{j}\right)\mathbf{j} + \left(\frac{d\alpha}{dP}\mathbf{k}\right)\mathbf{k}.$$

The reader must not be so unwary as to be led to believe that the components of the vector grad α are the parentheses. The parentheses are linear vector functions which arise from applying the derivative of a homography to a vector, and the vector grad α is that which results from operating with these homographies upon the vectors after the parentheses. Although the definition of grad α is thus given in terms of the system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the vector is really independent of any set of axes. It is unfortunate that an absolute definition like those of (1), (2), (3) was not given. It looks as if the authors had temporarily fallen back to some extent into the fatal slough of tachygraphy against which they are so careful to warn us. They go on to remark that in case the homography α reduces to a number m, the noteworthy result

$$(\operatorname{grad} m) \cdot dP = dm, \quad \text{Cf. (1)},$$

arises; and with delightful ingenuousness they add that this relation does not appear to have a correlative for other homographies than numbers. Of course not! If the authors had seen fit to call the vector defined by (10) curl α or $\sqrt{\alpha}$ or anything else selected at random from the vast realm of mathematical notations, they might have observed a similar lack of analogy. There is only one exception — if they had called their vector div $K\alpha$, they would have been surrounded on every side with the most persistent analogies.

Now after seeing the authors complain so much more bitterly than anybody else about the chaos of present vectorial notations and protest so vigorously against the use of a single symbol for the laplacian operator, it is difficult to say whether it is amusing or depressing to see these same authors introducing the old familiar symbol grad in a sense which they naively admit has small analogy with its former significance. Do they imagine that their present selection of notation will alleviate the chaotic condition? And do they find that the use of the same symbol for unrelated things is justifiable when it is not for things essentially identical?* If they had but listened to themselves half so attentively as they would have old hands at vector analysis listen to them, they would at least for safety's sake have used some other symbol than grad for (10) when they discovered that (10) had practically no connection with the well known grad. For this reason their choice of grad cannot be attributed to carelessness. It must be attributed to deliberateness. And this in face of the fact that various persons have used the term div in essentially the same sense as the authors use grad. Especial mention may be made of the extremely incisive and suggestive remarks of Prandtl† who treats the very subject of elasticity for which the authors are most in need of this new symbol. Even if an astute logician found no objection to the use of grad for div, it would seem as though a profound student of elasticity must feel intuitively that the forces in an elastic body arise from the divergence rather than from the slope of the fundamental homography connecting the normals to plane areas with the pressures upon them.‡

* It is interesting to quote from the authors, *L'Enseignement*, p. 466: "En conclusion, peut-on admettre, dans les mathématiques, un même nom, un même signe, pour indiquer deux choses différentes? Nous ne le croyons pas; par conséquent *nous n'avons pas suivi et nous ne suivrons jamais cette voie, qui conduit inévitablement à faire des confusions.*" The italics are ours.

† *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 13, pp. 436-449. This keen article should have the careful attention of all who are interested in vectors.

‡ The true inwardness of the incidental relation $dr.\nabla m = dm$ which arises when m is a numerical homography is seen by writing m in its proper form as mI where I is the idemfactor. Then $\nabla \cdot (mI) = \nabla m. I + m\nabla \cdot I$. But as I is constant, $\nabla \cdot I = 0$ and $\nabla \cdot (mI) = \nabla m$. Perhaps if the authors had not confused m and mI they would not have been led to mistake their relation grad $m \cdot dP = dm$ for an analogy with the gradient formerly defined.

Although this review is becoming lengthy we may perhaps be allowed the space to indicate what is probably the logical method of procedure in treating variable strains. With slight modifications we might use the authors' own excellent definitions (1), (2), (3) and write

$$(1) \qquad d\mathbf{r} \cdot \nabla u = d\mathbf{r} \cdot \text{grad } u = du,$$

$$(2) \qquad d\mathbf{r} \times \delta\mathbf{r} \cdot \text{curl } u = \delta\mathbf{r} \cdot du - dr \cdot \delta u,$$

$$(3) \qquad \text{div } u = \mathbf{a} \cdot [\text{grad } (\mathbf{a} \cdot u) + \text{curl } (\mathbf{a} \times u)],$$

where in (1) the entity u might be a scalar, vector, linear vector function, etc., and where in (2) and (3) the entity u may be a vector, linear vector function, etc., but not a scalar. With these definitions the reader may readily show the relation between div α and what the authors call grad α. If the methods of section 4 are adopted and extended, the definitions of curl and div become dependent on that of ∇ and the last two definitions may be suppressed. In fact these last two are introduced in the first instance so that the curl and divergence of a vector may be defined without introducing the gradient of a vector which would be a linear vector function. In like manner these definitions may be retained and the curl and divergence of a linear vector function may be defined without using the gradient of a linear vector function, which is an operator that converts vectors into linear vector functions. It is interesting to note that from the point of view of double multiplication (which must be considered in any general theory of Lückenausdrücke) the definition of the curl may be written

$$d\mathbf{r} \times \delta\mathbf{r} \cdot \nabla \times \Phi = (d\mathbf{r}\delta\mathbf{r} - \delta\mathbf{r}d\mathbf{r}) : \nabla \Phi$$

and becomes formally identical with the equation

$$\alpha \times \beta \cdot \gamma \times \Phi = (\alpha\beta - \beta\alpha) : \gamma\Phi, \quad (\alpha, \beta, \gamma \text{ vectors}),$$

which is a special case of a very general relation.

8. It will not be feasible to give any account of the last chapter of the Omografie, which contains applications to elasticity and to electrodynamics. There remains no space for such comments and there is very little to say. In bringing this review to a close it should be stated that the preponderating length of our adverse criticisms must not be interpreted as a wholesale condemnation of the two volumes. It has doubt-

less been noticed that the criticisms have been directed against those particular points at which, we feel confident, the authors have made incorrect statements or have unwisely abandoned fruitful algorisms and have thereby left the reader with a wrong or an unfortunately restricted point of view. There is no need to emphasize the excellent features of the work. A large number of these features are common to a considerable number of previous texts on vector analysis; many of them are new. The fact that there are so many points in which the volumes do not meet our approval is in itself evidence of the value of the books to all students of vectorial methods. In order to acquire a thorough appreciation of a subject it is necessary to examine various methods and points of view, and the restricted or even the wrong ones furnish an amount of instruction which is comparable with that furnished by those that are general and right. The present advanced state of the theory of functions of a real variable is due in no small measure to the inaccuracies or the narrow vision of earlier writers, and a considerable amount of present day instruction in this subject goes to showing that which is not true and contrasting it with that which is true.

For the benefit of vector analysis and cognate fields of mathematics we sincerely urge the general study of this work of Burali-Forti and Marcolongo and we especially recommend that each student follow their example and construct the system that pleases him most. That will be the best possible monument to the movement for unification. It will accomplish a real unity of knowledge and out of the resulting incidental diversity there will come a general and perchance not very slow elimination of the less fit and selection of the more fit. What the resulting residual system may be we will not venture to predict, but that there will be such a system fifty years hence we fully believe. And whatever that system may be it should and probably will conform to two requirements:

1° *Correct ideas* relative to vector fields,

2° *Analytic suggestiveness* of notation.

The first requirement may be fulfilled by proper teaching regardless of notation, whether vectorial, quaternionic, or cartesian; for the physicist the second is perhaps now best exemplified by the system of Gibbs, but the future may develop something preferable.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
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