From equation (6) it is seen that the coefficients of this polynomial,

$$\frac{z(1-z)^k}{s^{k+1}} = \frac{s^{n-k-1}MX}{(1-z)^{n-k}} \qquad (k = 1, \cdots, n-1),$$

are expansible as power series in $x_1, \cdots, x_p$, with positive coefficients, vanishing for $x_1, = \cdots = x_p = 0$. Hence the theorem is proved.

UNIVERSITY OF CHICAGO,
August 14, 1910.

KOWALEWSKI'S DETERMINANTS.

Einführung in die Determinantentheorie einschliesslich der unendlichen und der Fredholmschen Determinanten. By GERHARD KOWALEWSKI. Leipzig, Veit & Comp., 1909. 540 pp.

A PRELIMINARY survey of the contents of this book may best be obtained by dividing it into three nearly equal parts. The first of these parts comprises Chapters I–X and deals with the pure theory together with the single application to systems of linear equations * — a subject both historically and logically so intimately connected with the theory as to be almost inseparable from it. In the second part — Chapters XI–XVI — certain applications to algebraic, analytic, and geometric problems are treated. The third part consists of Chapters XVII–XIX and deals with two extensions of the idea of determinants obtained by allowing the order of the determinant to increase indefinitely. Both the theory of these determinants (infinite determinants and Fredholm determinants) and the closely related theory of the corresponding generalizations of systems of linear equations (linear equations with an infinite number of variables and linear integral equations) are treated in these chapters. While the introduction of such subjects into a book on determinants is not wholly unprecedented — Mathews' revision of Scott's book containing a brief discussion of infinite determinants — it must still, in view of its extent, be regarded as an innovation. This feature of the book is to be welcomed and will doubtless be imitated by other writers of text-books on this subject.

* A few equations of higher degree (secular equation, etc.) which are intimately connected with the theory of determinants are also considered in this first part.

We will now take up the chapters briefly one by one:

Chapter I (5 pages) is an interesting historical introduction in which translations are given of a passage in a letter of Leibnitz containing a definition of determinants, and of the note by Cramer where his rediscovery of determinants and his rule for solving linear equations were first published.

Chapter II (17 pages) gives, after a rather elaborate preparation for the specification for the signs of the terms of a determinant, the general definition with applications to two and three rowed determinants.

In Chapter III (9 pages) the very simplest properties of determinants (interchange of two rows, multiplication of the elements of a row by a constant, etc.) are deduced; and at the close it is shown how certain of these properties characterize the determinant completely. This fact is not used in later chapters, however, as Kronecker has shown that it can be used.

Chapter IV (13 pages) is entitled Sub-Determinants and contains Laplace's development with the development according to the elements of a row as a special case. It closes with a treatment of Vandermonde's determinant, that is the determinant of the nth order whose ith row is

$$a_i^{n-1},\ a_i^{n-2},\ \cdots,\ a_i,\ 1.$$

In Chapter V (19 pages) the theory of linear algebraic equations is treated. It begins with a proof of Cramer's Rule. Then after the conception of rank has been introduced and a few simple theorems concerning it have been proved, the subject of linear dependence is discussed. Then systems of *homogeneous* equations are taken up at length; and finally the non-homogeneous case is treated.

Chapter VI (13 pages) is entitled Multiplication of Matrices and Determinants. Here, and constantly from this point on, the following notation and terminology are used: If

$$x_1,\ x_2,\ \cdots,\ x_n, \qquad y_1,\ y_2,\ \cdots,\ y_n$$

are two systems of n quantities each, the quantity

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

is called their inner product (or simply their product) and denoted by (xy). This introduction of an idea of Grassmann is a distinctly good thing. It is, however, to be regretted that either here or at some other early stage of the book the subject were not treated in a somewhat more extensive manner by introducing the term *vector in space of n dimensions*, the conception of the sum of two vectors, the norm of a vector, the orthogonality of two vectors, etc. All of these conceptions are used freely in Chapter XVII in the case of space of an infinite number of dimensions, but before that point, where we have to deal with space of a finite number of dimensions, the term vector is not used at all, and such of these conceptions as are introduced occur in the midst of special investigations, for instance orthogonality is first explicitly mentioned on page 276 in an investigation concerning quadratic forms, although the idea (and also the conception but not the term *norm*) was really used in Chapter X. This is an illustration of the lack of complete coördination between the different parts of the book to which we shall have occasion to return later.

As a generalization and at the same time an application of this conception of the inner product we next have the product of two matrices

$$\left\|\begin{matrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} \end{matrix}\right\|, \qquad \left\|\begin{matrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ b_{m1} & \cdots & b_{mn} \end{matrix}\right\|,$$

which is defined as the *determinant* of the mth order whose general element (ith row and jth column) is the product, in the sense just explained, of the ith row of the first matrix into the jth row of the second. It is proved that the product, in this sense, of two square matrices is equal to the product of their determinants (so that the theorem of multiplication of determinants is established), that when $m > n$ the product is zero, and that when $m < n$ it is the sum of the products of all corresponding m-rowed determinants in the two matrices. As an elegant application for the law of multiplication of determinants a formula for the square of Vandermonde's determinant is given which is essentially the expression for the discriminant of an algebraic equation in terms of sums of powers of the roots.

Chapter VII (33 pages) is entitled Determinants whose Elements are Minors of Another. It opens with the familiar theorems about the determinant whose elements are the cofactors of another, and concerning the minors of this determinant. Some theorems of Sylvester of a more complicated character follow; and a treatment of bordered determinants is also included.

Chapter VIII (21 pages) on Symmetrical Determinants contains, besides a discussion of several special symmetrical determinants, an interesting proof that a symmetrical determinant of rank r contains at least one non-vanishing principal minor of order r, and a treatment of the secular equation, that is the equation obtained by equating to zero a determinant derived from a symmetrical determinant with real constant elements by adding the variable x to each element of the principal diagonal. The last section refers to a generalization of this equation which bears the same relation to a Hermitian matrix (that is to a matrix in which a_{ij} and a_{ji} are conjugate imaginary quantities) that the secular equation bears to real symmetric matrices. Some properties of Hermitian matrices are also briefly established, but the name *Hermitian*, which is freely used later in the book, is not mentioned.

In Chapter IX (27 pages) on Skew Symmetrical Determinants the following two fundamental theorems are first proved: *first* that a skew symmetric determinant of odd order is always zero, and *secondly* that a skew symmetric determinant of even order is the square of an integral rational function of its elements. This integral rational function, known as the Pfaffian, is studied, and its close analogy in theory and application to a determinant is brought out. The rank of a skew symmetric determinant is then considered; and finally a number of special determinants connected with this subject are treated.

Chapter X (19 pages) is on Orthogonal Determinants. After the fundamental properties of these determinants have been established, some special theorems of Brioschi, Siacci, and Stieltjes are proved, among which the first theorem of Siacci is especially noteworthy both for itself and for the application made of it to a simple proof of Cayley's formulæ. These formulæ, by means of which the elements of an orthogonal determinant are expressed rationally by means of $\frac{1}{2}n(n-1)$ parameters, are here proved on the supposition that the determinant has the value $+1$ and that the determinant obtained by adding 1 to each element of the principal diagonal is not zero. The formulæ are worked out in detail for determinants of the second and third orders.

A footnote at the beginning of this chapter tells us that the study of orthogonal determinants shall here be restricted to the case in which all the elements are real. This seems at first quite incomprehensible since, except at one or two points, where the restriction might easily have been explicitly made, the results and methods of the chapter apply equally well to the case of complex elements. The reason for the restriction to real elements becomes evident only at a much later point in the book (page 282) where, in the course of a special investigation on Hermitian forms, the word *orthogonal* is applied to the case of complex elements when the quantity

$$a_{i1}\bar{a}_{j1} + a_{i2}\bar{a}_{j2} + \cdots + a_{in}\bar{a}_{jn}$$

has the value 1 or 0 according as i and j are equal to or different from each other, — the a's being the elements and dashes being used to denote the conjugate complex quantity. While such a departure from well established usage in terminology can hardly be countenanced, it is certain that we have here a conception of importance * which under some other name is well worthy of place in a book on determinants. It is only to be regretted that it was not treated in a somewhat systematic manner in the chapter on orthogonal determinants.

We pass now to what we have called the second part of the book, — Applications.

Chapter XI (32 pages) on Resultants and Discriminants deals exclusively with binary forms. Sylvester's dialytic method (supplemented so as to be complete), Bézout's method, the relation of resultant and discriminant to the linear factors of the forms, finally their invariant character together with a few words about invariants of binary forms in general are some of the main subjects treated. The treatment at some points, for instance in §§ 78, 79, is remarkably elegant and lucid.

Chapter XII (43 pages) on Linear and Quadratic Forms opens with a consideration of the rank of a system of linear forms and, in case the number of forms is equal to the number of variables, their resultant regarded as invariants with regard to linear transformations. Then comes a brief treatment of bilinear forms in which, for the first time, that product of two matrices is considered which is itself a matrix. The remainder

* Cf. Frobenius, *Crelle*, vol. 95 (1883), p. 267 ; E. Study, *Math. Ann.*, vol. 60 (1905), p. 321, and I. Schur, *Math. Ann.*, vol. 66 (1909), p. 489, where references to some further literature will be found.

of the chapter is devoted to quadratic forms, their reduction to sums of squares, the law of inertia and the classification of real quadratic forms, the adjoint form, the rational invariants of pairs of quadratic forms, etc. Here, with slight exceptions, both the facts proved and the methods of proof are identical with those contained in the reviewer's Introduction to Higher Algebra.

Chapter XIII (36 pages) is on Elementary Divisors, and gives, in very brief space, the fundamental theorems on the equivalence of pairs of bilinear or quadratic forms. The method used, which is due to Stickelberger, depends, as did Weierstrass's original method, on a reduction to normal form, and like that method it involves unnecessary irrationalities. It depends on analytic tricks for which no motive is apparent, though the author has met this last objection to some slight extent by a similar, though much simpler discussion for a single quadratic form in § 106. Its compensating advantage, apart from its brevity, is that, like Weierstrass's original method, it gives a practicable method of approach to the subject of real quadratic forms, a subject in which, however, only a very special case is here considered.

The last part of this chapter contains an interesting exposition of the real *orthogonal* reduction of a single real quadratic form to a form in which only square terms enter, together with a generalization to the case of Hermitian forms. This work does not involve the theory of elementary divisors in any way, so that the place which has been given it, at the end of the chapter on elementary divisors, is unfortunate as many readers without the time or inclination to make themselves familiar with this theory will be likely to miss altogether this elegant treatment.

Chapter XIV (30 pages) on Functional Determinants opens, after the conception of the Jacobian has been defined, with the remark that the Jacobian

$$\frac{d(u_1, \cdots, u_n)}{d(x_1, \cdots, x_n)}$$

can be written as the quotient

$$\begin{vmatrix} d_1u_1 & d_1u_2 & \cdots & d_1u_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ d_nu_1 & d_nu_2 & \cdots & d_nu_n \end{vmatrix} : \begin{vmatrix} d_1x_1 & d_1x_2 & \cdots & d_1x_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ d_nx_1 & d_nx_2 & \cdots & d_nx_n \end{vmatrix}$$

where $d_i u_j$ denotes the differential of u_j which corresponds to the increments

$$d_i x_1,\ d_i x_2,\ \cdots,\ d_i x_n,$$

and where the increments of the x's are chosen arbitrarily but so that their determinant is not zero. It is then shown that, under suitable restrictions, the Jacobian can be obtained as the limit of a similar ratio where the differentials of the u's are replaced by increments. It would have been interesting for the reader to have been told whether this result is of any importance, and if so why.

There follows the law of multiplication for functional matrices and determinants. The transformation

$$\begin{aligned} \xi_1 &= u_1(x_1,\ \cdots,\ x_n), \\ \xi_2 &= u_2(x_1,\ \cdots,\ x_n), \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \xi_n &= u_n(x_1,\ \cdots,\ x_n), \end{aligned}$$

when the Jacobian of the u's is not zero, is then studied for the neighborhood of individual points, the work being carried through in detail for the case $n = 2$. Then comes a treatment of the effect of the rank of the functional matrix on the number of independent functions in the neighborhood of a non-singular point. Finally inverse functions, implicit functions, and the minors of the Jacobian are taken up.

Chapter XV (17 pages) on Wronskian and Gramian Determinants is one of the most novel and timely in the book. If $f_1(x),\ \cdots,\ f_n(x)$ are continuous (or more generally if these functions and their squares are integrable), but not necessarily real, throughout the interval $a \leqq x \leqq b$, we understand by their Gramian the determinant of the nth order in which the term in the ith row and jth column is

$$\int_a^b f_i(x)\bar{f}_j(x)dx,$$

the dash indicating the conjugate complex quantity. The vanishing of this determinant is a necessary and sufficient condition that $f_1, \cdots, f_n$ be linearly dependent.* If $f_1, \cdots, f_n$ are linearly independent, their Gramian is always positive. On the other hand, if

$$a_{i1}, \; a_{i2}, \; \cdots, \; a_{ik} \qquad (i = 1, 2, \; \cdots, \; n)$$

are n sets of k constants each, we understand by their Gramian the determinant of the nth order in which the element in the ith row and jth column is

$$a_{i1}\bar{a}_{j1} + a_{i2}\bar{a}_{j2} + \cdots + a_{ik}\bar{a}_{jk}.$$

Here again the vanishing of this determinant is a necessary and sufficient condition for the linear dependence of the sets of constants; and if these sets are linearly independent, their Gramian is positive. We have here a striking analogy, but unfortunately no attempt is made in the book under review to explain *why* this analogy exists. Such an explanation would not have been difficult, for instance one may regard the case last mentioned as a special case of the earlier one by letting $a = 0$, $b = k$ and

$$f_i(x) = a_{ij} \quad \text{when} \quad j - 1 < x < j$$
$$(i = 1, 2, \cdots, n; j = 1, 2, \cdots, k).$$

Or one may take the more general point of view of considering n sets of k *functions* each

$$f_{i1}(x), f_{i2}(x), \; \cdots, f_{1k}(x) \qquad (i = 1, 2, \; \cdots, \; n)$$

continuous throughout the interval $a \leqq x \leqq b$, or, more generally, such that these functions and their squares are integrable throughout this interval. By their Gramian we then understand the determinant of the nth order in which the element in the ith row and jth column is

$$\int_a^b [f_{i1}(x)\bar{f}_{j1}(x) + \cdots + f_{ik}(x)\bar{f}_{jk}(x)]dx,$$

*If $f_1, \cdots, f_n$ are not assumed to be continuous, we understand here by linear dependence throughout the interval $a \leqq x \leqq b$ the existence of n constants $c_1, \cdots, c_n$ not all zero and such that

$$c_1 f_1 + \cdots + c_n f_n$$

vanishes at every point of this interval except at the points of a set of content zero.

and here again this Gramian will always be positive except when the n sets of functions are linearly dependent, in which case it vanishes. This conception clearly includes the two above mentioned as special cases.

However much the chapter under consideration is to be welcomed, it is to be regretted that it had not been made more complete. This might have been effected in part by collecting here various discussions which are scattered through the later chapters and which really refer to this subject; for instance the proof of Hadamard's inequality for the absolute value of a determinant on pages 458–460. This inequality is simply a corollary to the theorem that a Gramian (in any of the senses above mentioned) can never exceed the product of the elements of its principal diagonal.* Again the relation of Gramians to definite Hermitian forms is not considered in this chapter.

One section of this chapter is devoted to the relation of Wronskians to the subject of linear dependence, but after three and a half pages of this discussion the reader is left without the knowledge of the most important, and one of the most easily proved theorems in this subject, namely that if n functions are analytic, the identical vanishing of their Wronskian is a sufficient condition for linear dependence.

In comparing the Wronskian with the Gramian test for linear dependence the important fact is pointed out that while the former test requires, in the case of n functions, the existence of the first $n - 1$ derivatives, for the latter the continuity of the functions is sufficient. The even more important fact that the Gramian test applies to functions in any number of variables is not mentioned.

Chapter XVI (32 pages) is entitled: Some Geometrical Applications of Determinants. More than half of it is devoted to applications in the domain of elementary analytic geometry: The area of a triangle in terms of the coördinates of its vertices and in terms of the equations of its sides; the product of the areas of two triangles in a plane in terms of the distances between their six vertices; a similar result in case the two triangles are inscribed in a circle of radius r; the characteristic relation between the distances of four points on a circle; finally the generalization of all of these results to space. Then come two

* If none of the functions (or systems of functions) are identically zero, it reaches this upper limit when and only when the functions (or systems of functions) are mutually orthogonal.

sections on affine transformations in the plane, with special reference to the fact that all areas are changed in the same ratio. Finally the geometrical interpretation of the Jacobian is discussed of which the property of affine transformations just mentioned is a special case, the work being here again confined to the neighborhood of a non-singular point.

* * *

We pass now to the third part of the book.

Chapter XVII (86 pages): Determinants of Infinitely High Order, might well have been split into two chapters. It should at least have been clearly indicated that it consists of two parts wholly independent of each other. The first part, somewhat the shorter of the two, closes on page 407. If we have a doubly infinite array of quantities

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}$$

and if we form from them the determinant of the nth order

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix},$$

it may happen that this determinant has a finite limit when $n = \infty$. This limit is then called a determinant of infinitely high order and is denoted by putting vertical bars on the two sides of the infinite array above. Our author considers primarily *normal* determinants, that is determinants for which the double *series*

$$\sum_{i,j=1}^{\infty} c_{ij}$$

converges where

$$c_{ii} = | a_{ii} - 1 |, \quad c_{ij} = | a_{ij} | \qquad (i \neq j).$$

Such determinants are necessarily convergent as are also their minors and all determinants obtained from them by replacing a finite number of their rows or columns by quantities which are in absolute value less than some constant.

If in the infinite system of linear equations with an infinite number of variables

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots = b_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

the a_{ij}'s are such that the infinite determinant formed from them is a normal determinant, and the b's do not exceed in absolute value some constant, it can be shown that normal determinants and their minors can be used for the solution of the system precisely as finite determinants can be used for the solution of finite systems of linear equations with a finite number of variables. This, together with the extension of numerous properties of finite determinants to infinite normal determinants, forms the contents of this first part of the chapter.

The second part contains an exposition of E. Schmidt's theory of systems of linear equations, and *does not deal with infinite determinants at all.* It is true that determinants occur here whose order n increases indefinitely, but these determinants do not in general themselves approach finite limits when $n = \infty$ but only the ratio of two such determinants.* The subject has been made more attractive by the free use of the conception and language of vectors in space of an infinite number of dimensions. One misses, however, any statement of the relation of this method of Schmidt to the method of infinite determinants explained in the earlier part of the chapter. The relations would seem to be, in part, these:

1. Both methods deal with the solution of the infinite system of equations referred to above, and neither method treats the problem in all its generality.

2. In the method of infinite determinants it is assumed that the double series $\sum c_{ij}$ converges, while in Schmidt's method not even the convergence of the individual rows of this series,

* The determinants themselves can be made to converge by dividing the equations of the system we wish to solve by constants such as to make the series of the squares of the coefficients of the individual equations have values less than or equal to 1.

or, what amounts to the same thing, the convergence of the series

$$\sum_{j=1}^{\infty} | a_{ij} | \qquad (i = 1, 2, \cdots)$$

is demanded, but merely the convergence of the series

$$\sum_{j=1}^{\infty} |a_{ij}|^2 \qquad (i = 1, 2, \cdots).*$$

3. The method of infinite determinants yields, in the cases to which it is applicable, every solution $(x_1, x_2, \cdots)$ which is such that all the x_i's are in absolute value less than some constant, while Schmidt's method yields only those solutions $(x_1, x_2, \cdots)$ for which the series $\sum |x_i|^2$ converges.

4. When we use infinite determinants, we obtain the solution of the infinite system † as the limit for $n = \infty$ of the solution of the finite system

$$\begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n = b_1, \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n, \end{array}$$

that is we allow the number of equations and the number of unknowns to increase *simultaneously* always remaining equal. In Schmidt's method we start from the system

$$\begin{array}{c} a_{11}x_1 + \cdots + a_{1k}x_k = b_1, \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a_{n1}x_1 + \cdots + a_{nk}x_k = b_n \end{array}$$

and first, keeping n fixed, pass to the limit $k = \infty$; ‡ only after this has been done do we allow n to become infinite. In order to be able to carry this method through, new formulae must be obtained for the solution of the system of equations last written when $k > n$.

Some such explanation as this of the scope of the two methods and their relation to one another should have been given in the

* We pass over here the less easily stated conditions on the b's.

† At least in case the determinant of the system is not zero.

‡ This first passage to a limit is, in practice, avoided by treating *directly* the case $k = \infty$.

chapter under discussion. Instead of this, the two methods are merely placed side by side without comment. Moreover in the latter part of the chapter we miss any clear statements of the results obtained, so that it is almost impossible to find out precisely what Schmidt's method accomplishes without reading the greater part of the forty odd pages here devoted to it.

Chapter XVIII (50 pages) on Linear Integral Equations opens with a discussion of Fredholm's determinant. This, like the infinite determinant, is the limit of a determinant of the nth order as n becomes infinite; but while, in the case of the infinite determinant, the element in the ith row and jth column of the determinant of the nth order does not change with n when once n is greater than both i and j (just as in the case of an infinite series, which is the limit of the sum of n terms, these terms do not change with n), in the case of Fredholm's determinant the elements approach zero, or, in the case of the elements of the principal diagonal, one, as n becomes infinite (just as in the case of a definite integral, which is also the limit of a sum of n terms, each term approaches zero as n becomes infinite). To make this rough statement more precise, consider a function $f(x, y)$ continuous throughout the square

$$S \qquad 0 \leqq x \leqq 1, \quad 0 \leqq y \leqq 1.$$

Divide S into n^2 equal squares by lines parallel to the axes of x and y and denote the coordinates of the centres of these squares by

$$(x_i, y_j) \qquad (i, j = 1, 2, \cdots, n),$$

where the notation is such that both x_i and y_j increase as the subscript increases. Then, as n becomes infinite, the determinant

$$D_n = \begin{vmatrix} 1 + \frac{f(x_1, y_1)}{n} & \frac{f(x_1, y_2)}{n} & \cdots & \frac{f(x_1, y_n)}{n} \\ \frac{f(x_2, y_1)}{n} & 1 + \frac{f(x_2, y_2)}{n} & \cdots & \frac{f(x_2, y_n)}{n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{f(x_n, y_1)}{n} & \frac{f(x_n, y_2)}{n} & \cdots & 1 + \frac{f(x_n, y_n)}{n} \end{vmatrix}$$

approaches a finite limit called the Fredholm determinant of the function f for the region S.

About half of the chapter under consideration is devoted to the theory of these Fredholm determinants; their minors and relations between them, the product of two Fredholm determinants, etc., being discussed. The *results* are due to Fredholm; the *method*, that of passing to the limit from the algebraic case, was used by Fredholm as a heuristic one for discovering the facts. His method of proof, however, was a different and more elegant one. As a rigorous method of proof, this method was first used by Hilbert for establishing some of Fredholm's results, along with certain other results with which we are not here concerned. In other cases (the minors of higher orders) the chapter now under consideration seems to give the first exposition of this method of proof.

The other half of the chapter is devoted to the application of Fredholm's determinants to the integral equation of the second kind

$$\phi(x) + \int_0^1 f(x, y)\phi(y)dy = \psi(x),$$

where f and ψ are known functions, ϕ the function to be determined. Here the method used is that of Fredholm not of Hilbert, that is the integral equation is treated directly, not as the limit of a system of linear algebraic equations. The analogy with the case of linear algebraic equations is frequently mentioned and sometimes elaborated at some length, but we fancy this analogy will seem pretty obscure unless the reader is fortunate enough to notice the note on page 544, where the method of obtaining the integral equation as the limit of the algebraic system is explained and falsely attributed to Hilbert. It was known to Fredholm, and had been explained by Volterra in 1896.

Here, as in Chapter XV, it is unfortunate that no mention is made of the fact that methods and results apply without change to functions in any number of variables.

Chapter XIX (35 pages) is entitled: Hilbert's Characteristic Functions of a Real Symmetric Kernel, and gives, with certain interesting modifications of detail, some of the more important parts of E. Schmidt's dissertation (*Mathematische Annalen*, volume 63). It deals with the above written integral equation, chiefly in the case in which f is a symmetric function of (x, y), with special attention to the case in which ψ vanishes identically. While these developments of Schmidt are of the greatest importance and elegance, it is hard to see why they should be inserted into a book on determinants. More germane would seem to be Hilbert's theory of quadratic forms with an infinite number of variables, an exposition of which would have been, for other reasons also, most desirable.

* * *

Properly regarded, the theory of determinants is hardly more than a shadow of the theory of matrices, and just so far as one attempts to ignore this fact does the subject become an artificial one or, at best, a tool for other investigations. Every book on determinants must be dominated more or less by the conception of the matrix, which first appears as an inanimate object, a mere rectangular array of quantities from which determinants, whose vanishing or non-vanishing are observed or whose values are used, are cut out. Later, however, the matrix appears as an animated being capable of combination by addition and at least two kinds of multiplication with its kind, and it is in this aspect that the conception becomes a highly fruitful one. In the book under review the matrix as an inanimate object is freely used, and the less important kind of multiplication, according to which the product of two matrices is a determinant (cf. what was said in connection with Chapter VI) is adequately treated. That two matrices can be added together or that a matrix can be multiplied by an ordinary quantity (scalar) seems to be nowhere even mentioned. The extremely important kind of multiplication according to which the product of two square matrices of the nth order is another matrix of the nth order is not taken up until the section on Bilinear Forms in Chapter XII, that is, in what we have called the second part of the book, after the *theory* of determinants of finite order has been completed. Thus it will be seen that the idea of the matrix is used somewhat reluctantly. That the book nevertheless remains a good book is due largely to the fact that in many places where the word *determinant* is used a matrix is meant. A determinant is quite properly defined (page 19, near the top) as a *quantity* obtained by combining by the well-known rule the elements of a square matrix. On page 24 we have the theorem that a determinant is unchanged by an interchange of rows with columns. Later (page 161) comes the theorem that an orthogonal determinant remains orthogonal if rows are interchanged with columns. What a triviality if the earlier statement that every determinant remains unchanged was true!

The fact of course is that the theorem about orthogonal determinants is meant to refer not to a determinant, but to an orthogonal matrix. Indeed the whole chapter on orthogonal determinants is really a chapter on orthogonal matrices; the only fact proved, and the only non-trivial fact which can be proved about an orthogonal determinant (i. e., the determinant of an orthogonal matrix) being that it is equal to $+1$ or -1. All that we can mean by saying that a determinant (i. e., a quantity) is orthogonal is that an orthogonal matrix exists of which it is the determinant; and this is clearly true when and only when the determinant is $+1$ or -1. Consequently, to take one more example, the theorem that the product of two orthogonal determinants is an orthogonal determinant is a triviality — it merely says that if $a = \pm 1$ and $b = \pm 1$, then $ab = \pm 1$. What is *meant* is, of course, that the product of two orthogonal *matrices* is an orthogonal *matrix*, the kind of product meant being that which is defined only fifty pages later.

The advantage of the free use of the word and conception *matrix* would be not merely to avoid calling two different things by the same name, perhaps even in a single sentence, — a most pernicious thing in mathematics — but also to permit much briefer and more transparent formulations of many proofs. For instance, it follows at once from the definition that a matrix A is orthogonal when and only when

$$A'A = I,$$

where A' is the conjugate (transposed) of A and I is the unit matrix

$$I = \left\| \begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{matrix} \right\|.$$

It follows that

$$A' = A^{-1}$$

and this is merely a more compact and elegant way of stating Theorem 43 on page 160. Since a matrix is always commutative with its reciprocal, we see that

$$AA' = I$$

and consequently A' is orthogonal. (Theorem 44, page 161.)

If A and B are both orthogonal

$$A' = A^{-1}, \quad BB' = I.$$

Multiplying these equations together gives

$$A'BB' = A^{-1}.$$

Consequently

$$A'BB'A = I.$$

But, $B'A$ being the conjugate of $A'B$, the orthogonality of $A'B$ is thus proved. (Theorem 45, page 161.)

The introduction of proofs of this character would presuppose a treatment of the fundamental facts in the algebra of matrices. The few pages necessary for this purpose would be well spent, as some of the most important papers published at present on determinants are written in part in the language of matrices.*

One of the strong features of the book is the admirable selection of subjects; for though, as we have pointed out, there are a few things which one would like to have seen added, there is very little which one would be willing to see omitted. Open the book where you will and you find yourself in the midst of a live question having vital connections at the present day with other live branches of mathematics. This is no small praise when one recalls the possibilities for barren formalism which the subject of determinants presents. The choice not only of subjects but of the special aspects of these subjects to be considered and of the individual theorems to be proved has been guided by a true sense of values, not by the mere love of formal developments.

It is therefore somewhat surprising that in the selection of methods of proof a tendency to give the preference to formal manipulations is frequently apparent. Someone has said: "C'est le sentiment de se mouvoir dans l'identique qui fait la joie de l'algébriste" — a statement which would be correct if the word *formaliste* were put in place of *algébriste*. The algebraist has no less pleasure in taking large steps where the

*Cf., for instance, the paper by I. Schur already cited.

feeling of identity is lost than has the analyst or the geometer; and indeed the criticism just made that many proofs in the book under review are of too purely formal a character is closely connected with the further fact that the author seems to make an effort to use as little algebra as possible. The conception of reducibility is nowhere used, and it is only in a brief note at the end of the volume (page 542) that the irreducibility of the general determinant is mentioned. Except in the case of the binary form, no reference is made to the fact that every polynomial can be resolved into irreducible factors in one and essentially only one way. Even such a fundamental fact as that a polynomial vanishes identically when and only when all of its coefficients are zero is not assumed to be known but is *proved* when, in the chapter on Elementary Divisors (page 258), it is found necessary.* Methods so thoroughly germane to the theory of determinants as are the elementary agebraic theorems just mentioned are avoided at the expense of simplicity and brevity. For instance, in the treatment of Vandermonde's determinant in Chapter IV, a formal proof covering two pages is given. This may perhaps be justified on the ground that at this early stage practice in applying the rules for manipulating determinants is desirable; but a satisfactory reason can hardly be given for the omission of the brief and lucid proof which says that the determinant, being a polynomial in $a_1, \cdots, a_n$ which vanishes when any two a's are equal, must have each of the differences $a_i - a_j$ as a factor. The other factor, into which the product of these differences is multiplied, which is clearly a constant, can then be easily determined.†

Another good illustration of the tendency in question is to be found in the proof that the determinant of a skew symmetric matrix of even order is the square of a polynomial in the elements of the matrix. Two proofs of this theorem are given, the first of which covers a page and a half, and the second (which is of a purely formal character involving an examination of the individual terms of the determinant) three pages. The theorem may be proved by mathematical induction, a method also used in the first of the two proofs just mentioned, as follows:

* Contrast with this the free use made, without explanation, of uniform convergence, the integration of series term by term, and various matters relating to the differential and integral calculus.

† We must use here the algebraic theorem (or at least an easily proved special case of it) which says that if a first polynomial vanishes whenever a second irreducible one does, the second is a factor of the first.

Denote the determinant of order $2n$ by D, its cofactors by A_{ij} and the determinant obtained by striking out its last two rows and columns by S. Then

$$SD = A_{2n-1,\,2n-1}A_{2n,\,2n} - A_{2n,\,2n-1}A_{2n-1,\,2n}.$$

But since $A_{2n-1,\,2n-1}$ and $A_{2n,\,2n}$ are skew symmetric determinants of odd order and therefore vanish, while

$$A_{2n,\,2n-1} = - A_{2n-1,\,2n},$$

the above formula reduces to

$$SD = A^2_{2n,\,2n-1}.$$

Consequently since S, being a skew symmetric determinant of order $2n - 2$, is by hypothesis the square of a polynomial in the elements, it follows, by an application of the theorem that a polynomial can be resolved into irreducible factors in essentially only one way, that the same is true of D.

The obvious reason for the second and longer proof is to introduce the subject of Pfaffians. This is, however, quite unnecessary. We may define the Pfaffian of a skew symmetric matrix of order $2n$ as the square root of its determinant, the sign being so determined that the Pfaffian reduces to $+1$ when

$$a_{12} = - a_{21} = 1,\; a_{34} = - a_{43} = 1,\; \cdots,\; a_{2n-1,\,2n} = - a_{2n,\,2n-1} = 1$$

while all the other a's are zero. This Pfaffian is then, by the theorem just proved, a polynomial of degree n in the a_{ij}'s the forms and signs of whose terms can be determined with ease by methods which make far less demand on the concentrated attention of the reader than is called for in the section under consideration, and which also have the advantage of much greater brevity.

We have already indicated by various examples that the relations of different parts of the book to each other are not always clearly brought out. A minor, though not unimportant illustration of this same weakness is to be found in matters of terminology. The definition on page 78 of the reciprocal determinant does not tally with the definition on page 241 of the reciprocal quadratic form. The definition on page 49 of the rank of a finite determinant (or more accurately matrix) does not tally with the definition on page 399 of the rank of an infinite determinant, nor with the definition on page 484 of the rank of a Fredholm determinant. Connected with a matrix of order n are two integers, its *rank* and its *nullity*.* The sum of these two numbers is n, so that we may dispense with either one of them, and most authors use the rank alone. It is, however, sometimes convenient to use the nullity, and it is the generalization of this latter conception, not of the rank, with which we have to deal in the case of infinite determinants and of Fredholm determinants.

A more detailed table of contents containing section as well as chapter headings, and a more extensive index would have been of much assistance to users of the book.

Many will be inclined to criticise the scantiness of the bibliographical references placed, along with a few other notes, at the end of the volume. It must be remembered, however, that the collection of *reliable* information of this sort for such a widely ramified and, in the main, old field as that here treated is a very serious undertaking. An extensive and reliable bibliography would be, without doubt, a very welcome supplement, but it seems to be asking too much of the author of a book of this character to demand of him that he undertake such a task; the difficulties in producing good mathematical treatises are quite formidable enough without that.

We have said that the choice of subjects treated, both in the large and in detail, is admirable. The book has seemed to us at some points less satisfactory in the *methods* of proof selected, but these methods having once been chosen, the detailed exposition of the proof is always good and frequently a model of clearness and conciseness; it is evident that we have to deal with a writer whose expository powers are of the first order.

Last but not least, the standard of accuracy attained is remarkably high. In the sections which we have examined with care, and these constitute no inconsiderable portion of the book, the only error we have found is the one already mentioned of using the word *determinant* frequently in the sense of *matrix*, and this error is so sanctioned by universal usage that it must, for the present, be regarded rather as a defect of the age than of the individual. Only an author who has himself seriously tried to eliminate errors from his writings and is not satisfied, when a mistake is discovered, to say that he

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* The word *Defekt* has been used here in German.

wishes to be understood in some other sense, which his words do not admit (in other words to *crawl* if we may be permitted to use an expressive vulgarism) can appreciate the labor involved in the production of a text-book with this high record of correctness. We can recommend the book, in spite of some shortcomings, as one from which few readers can fail to get much valuable information without undue effort.

MAXIME BÔCHER.

DIFFERENTIAL INVARIANTS.

Invariants of Quadratic Differential Forms. By J. EDMUND WRIGHT. Cambridge Tract No. 9. Cambridge University Press, 1908. viii + 90 pp.

IN these days, when the number of papers in mathematics published each year is almost without limit and the ramifications are no less perplexing in their variety, one is delighted to find here and there a digest of the work in a particular field. These are the pleasures which the Cambridge Tracts hold in store for us, and we owe a debt of gratitude to our fellow-workers who are willing to pause in their researches to give us a panoramic view of their field and thus to turn over to us in nut-shell form the products of their searchings in the works of their colaborers. The author of the tract before us felt that this was its mission and he seems to have attained his hopes.

In his introduction he leads the reader into the domain of his subject by showing him an invariant and then defining it. The example taken is naturally enough the Gaussian curvature of a surface whose linear element

$$ds^2 = dx^2 + dy^2 + dz^2,$$

expressed in terms of two independent parameters u and v, is written

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

The idea of differential parameter is also suggested at this time. The reader is then acquainted with the magnitude of the field of inquiry and of the two kinds of problems: (i) The determination of all invariants of one or more differential quadratic forms and their relations; (ii) The geometrical and mechanical interpretation of these invariants.