jectories are the geodesics of the manifold, and they are described with constant velocity. An important result is that the knowledge of an algebraic integral of the system (12) carries with it that of a homogeneous integral of the differential equations for geodesics in the manifold. The chapter closes with a discussion of homogeneous linear and quadratic integrals of the equations of geodesics; the determination in invariant form of the criteria that two dynamical systems have the same trajections; and the geodesic representation of one manifold upon another.

The subject matter is presented in an inspiring way, so that it seems very probable that the reader will turn to the papers of Ricci and Levi-Civita, as the author hopes. The proof reading has been well done and in every way the printer's work is satisfactory.

It is impossible to close this review without remarking the loss to American mathematics by the death of Mr. Wright. His brilliant record at Cambridge and his subsequent career in this country had won for him a high place in his field.

LUTHER PFAHLER EISENHART.

STURM'S GEOMETRISCHE VERWANDTSCHAFTEN.

Die Lehre von den geometrischen Verwandtschaften. Vierter Band: die nichtlinearen und die mehrdeutigen Verwandtschaften zweiter und dritter Stufe. By RUDOLF STURM. Leipzig and Berlin, Teubner, 1909. x + 486 pp.

As the subject matter of this fourth volume of Professor Sturm's extensive treatise on geometric relations is so different from that of the preceding ones,* but little analogy can be drawn with the methods already discussed. With the exception of one elementary treatise, mathematical literature did not include a book on birational transformations before the appearance of the present volume, although over five hundred memoirs have been devoted to the subject during the last two decades. The newness of the subject, the possibility of approaching it from different standpoints, and its applicability to so many other

* These volumes have been reviewed in the BULLETIN; volume 1 in vol. 15, p. 135; volume 2 in vol. 15, p. 252; volume 3 in vol. 16, p. 250.

disciplines all contribute to make the task of preparing a systematic treatise both important and difficult. Some aspects must be omitted altogether and others but briefly sketched. As to the wisdom of any particular choice of material there will doubtless be a wide diversity of opinion, for methods that are of immediate use for one purpose are of no value for another, although both serve to develop the same theory. The present volume proceeds synthetically, with a slight digression into more algebraic methods in discussing the general linear transformations of a complex variable. In most cases a brief outline only is given, the author constantly referring the reader to the original memoirs for the details; over three hundred such references are supplied. As in the preceding volumes, a glossary of technical terms is provided, the present one containing 74 terms.

In the earlier volumes a number of illustrations of Cremona transformations were met; in some cases the image of a straight line was actually found, and the general theorem that the genus of a curve is invariant under any birational transformation of the plane was proved (volume 1, no. 163). The present volume begins with the problem of finding the most general net of curves that can be imaged on the ∞^2 straight lines of the plane. Several theorems regarding the number of constants which determine a curve, the intersections of curves, and the law of equivalence of multiple points are stated as lemmas; from these criteria and the fact that the curves are rational it follows that each curve must be fixed by two linear conditions and that any two curves intersect in one point apart from the basis points of the net. The discussion occupies 20 pages, contains all the fundamental relations, includes a complete table of forms of nets up to order 10, and is admirably well presented. The next ten pages are devoted to illustrations, containing a veritable mine of suggestions for further research. The chapter on quadratic inversion is unnecessarily long (33 pages) and detailed; the various forms of the principal elements are not kept sharply separate. Conformal representation and transformation by reciprocal radii receive a great amount of attention. At the end of this chapter the complex variable is introduced and the preceding theory is developed analytically. The involutorial (simple) inversions are consistently distinguished from general non-periodic ones, but it is not shown that the latter can be obtained from the former by multiplication with a collineation.

Involutorial transformations of higher order are now discussed, largely following Doehlemann, but the Geiser transformation, fixed by a net of general cubic curves having seven distinct basis points, is also treated.

The first chapter closes with the theory of correspondence applied to ternary fields. The general formula for the number of coincidences is derived, then confirmed by numerous illustrations, taken from cases in which the number of coincidences is determined in another way.

The second chapter is devoted to correspondences on curves of genus one. Most of it is confined to the general plane cubic, then the space quartic of the first kind, the plane binodal quartic and the ruled quartic surface having two double directrix lines are considered. Each of these entities can be rationally mapped on any one of the others. The treatment is essentially that of Weyr, and is very complete, containing a full geometric theory of the properties of cubic curves and a well drawn picture of the configurations which define finite Cremona cycles, as well as of the G_{18} of collineations under which a non-singular, cubic curve remains invariant.

The problem of multiple correspondences between ternary fields is again taken up in an instructive chapter which considers in detail the (1, 2) case. The procedure is analogous to that of Paolis, but the latter is only an abstract of the wealth of material and particularly of new view points here developed. If the genus of the curves in the simple field which are the images of the straight lines in the double field is called the genus of the transformation, it is shown that we need only consider the two cases, genus zero and genus one. The basis points, the coincidence curve of the involution in the simple field, and the contact curves of the curve of branch points in the double field are exhaustively treated. A typical case is the Geiser transformation, already discussed in the preceding chapter. A later section considers (m, 1) and (2, 2) correspondences. Finally, a number of illustrations are added to show the usefulness and power of the ideas here developed. An immense field of uncultivated territory is opened by this discussion.

As a suitable transition from Cremona transformations in the plane to those in space, a chapter is introduced which treats of those surfaces which can be rationally mapped upon the plane. The general stereographic projection of a quadric surface is followed by a much fuller discussion of the case of the cubic. Two general methods are considered, that of trilinear pencils of planes and that of the linear congruence whose directrices are two skew lines lying on the surface. This is followed by the Steiner surface, the cubic ruled surfaces, the quartic having a double conic and the quintic having a double space cubic. Finally, the depiction of a surface of order n which has an $n-2$-fold line and the Noether surface of order 4 are discussed. One interesting and important question in this connection is hardly touched upon, namely, to determine the nature of those transformations in space which go into Cremona transformations in the plane under which a plane curve remains invariant.

The birational transformations of space are introduced by making a system of surfaces having four homogeneous linear parameters collinear with the planes of space. The images of lines in one space and of planes in the other are shown to be of the same order, but those of planes in the first and of lines in the second are of an order independent of the first number, being fixed by the configuration of the principal elements. In order that the operations be reversible, the principal elements must absorb all but one of the intersection of any three surfaces of the system. The principal points, curves, and surfaces are brought in, together with the principal curves of the second kind, i. e., the locus of points whose images are all the same principal curve. The entire configuration is shown to constitute the Jacobian of the system. The images of lines in either system must be rational curves, and of planes must be homaloidal surfaces.

This short introduction is then amplified by studying various particular cases in detail, the first being a system of quadrics passing through four fixed points and touching a fixed plane at one of them. The system in the other space is composed of Steiner surfaces. When the quadrics have a line in common, the Steiner surfaces are replaced by cubic ruled surfaces. If the quadrics have a conic in common, the other system is also composed of quadrics. A particular form of this case is the transformation by reciprocal radii, which is discussed at some length. The second illustration is that of general cubic surfaces. It begins with Meusnier's theorem regarding the curvature of plane normal sections of a surface, in order to provide for an intersection of two cubics of the desired form. On account of the number of forms of the cubic surface this case receives a long discussion. An extensive knowledge of the forms of cubic surfaces is presupposed, the author constantly referring to his own book on this subject.

The chapter on involutorial transformations is an enumeration of a number of the better known cases, such as generalized quadratic inversion, the polarity of a cubic surface from a point upon it, the harmonic conjugate of a point with regard to the points of intersection of a line through it and cutting a given cubic curve twice, and the pairs of associated points defining a bundle in a system of ∞^3 quadric surfaces passing through six fixed points, this last case being the Geiser transformation of space. The peculiarities of the fifteen lines joining the six points by twos and the cubic curve passing through all of them are discussed in detail. These lines are interesting as furnishing the first example of a principal curve of the second kind. The more general investigations of Montesano are not considered in the present volume.

The last chapter treats of multiple correspondences in space of three dimensions. It begins with the determination of the number of coincidences, and confirms the result by numerous simple illustrations. A short discussion is devoted to correspondences in line space; the general formula for the number of coincidences is derived and a simple illustration given. As in the case of the plane, the next section considers more in detail the (2, 1) correspondence first studied by Paolis. Most of the results are directly analogous in space to those obtained above for the plane, but to follow the proofs a knowledge of the author's treatise on line geometry is necessary. This interesting section is followed by two cases of (2, 2) quaternary correspondences, and a generalization of the duality defined by a linear complex (höhere Nullverwandtschaft). A short appendix completes the proofs of a few theorems in the preceding volumes and extends a few results to more general cases.

VIRGIL SNYDER.

NOTES.

THE Annual Register of the AMERICAN MATHEMATICAL SOCIETY is now in preparation and will be issued in January. Blanks for furnishing necessary information have been sent to the members. Early notice of any changes since the issue of the last Registerill wgreatly facilitate the work of the Secretary. The