graduate courses and the relative frequency of their occurrence. In the present paper a similar study is made of the calculus used by the practicing engineers, the more recent volumes of leading technical journals furnishing the data used.   He discusses also the emphasis placed on calculus and the attitude of the engineers of different nationalities toward it.

12. Mr. Smail considers a hypercomplex number system in two units e_1, e_2, where $e_1^2 = e_1$, $e_1 e_2 = e_2 e_1 = 0$, $e_2^2 = e_2$.   He considers the elementary conceptions and the functionality of the simpler algebraic expressions.   Differentiation is defined and its limitations pointed out; integration is defined and the integral of a hypercomplex variable expressed in terms of ordinary curvilinear integrals.   He finds also the conditions that make the integral independent of the path of integration.

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H. C. MORENO,
Secretary of the Section.

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INVARIANT CONDITIONS THAT A p-ARY FORM MAY HAVE MULTIPLE LINEAR FACTORS.

<div align="center">

BY PROFESSOR O. E. GLENN.

(Read before the American Mathematical Society, October 29, 1910.)

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§ 1. *Introduction.*

IN the case of a ternary quadratic form the vanishing of the ordinary discriminant is the necessary and sufficient condition that the form be factorable into linear factors, distinct term for term.   In the case of a cubic ternary form we can obtain in various ways a set of three independent rational, integral, and homogeneous functions of the coefficients, whose simultaneous vanishing furnishes the necessary and sufficient conditions that the form be factorable into distinct linear factors.   The general theorem underlying these facts has been proved by Junker, * and is that the necessary and sufficient conditions that a p-ary form of order m be a product of m distinct linear factors consist in the simultaneous vanishing of a set of $\binom{m+p-1}{m} - m(p-1) - 1$ independent seminvariant † relations among the form's coefficients.

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* Junker, "Die symmetrischen Funktionen u. s. w.," *Math. Annalen*, vol. 45.

† Junker, "Ueber die Diff.-Gleichungen der Invarianten," loc. cit., vol. 64.

Several papers * have been published on the problem of determining such sets of conditions. Brill † and Gordan ‡ have each treated the problem by the symbolical methods, obtaining in each case a redundant set of conditions furnished by the identical vanishing of a covariant. Junker confined his attention to the problem of finding in terms of the coefficients the $\binom{m+p-1}{m} - 1$ identical relations which exist among the elementary symmetric functions of the m groups of p homogeneous variables of the form. His sets of relations also do not contain the minimum number. There exist $m(p-1)$ syzygies among them.

Not much attention seems to have been given however to the problem of determining a minimum set of conditions that a p-ary form $f_{pm}(p > 2)$ may possess *multiple* linear factors. It is the purpose of this note to show how a certain factor theorem for f_{3m}, which we give in the first part of § 2, may be made the basis of various methods of determining such sets. In § 2 we determine conditions in order that a p-ary m-ic f_{pm} should be a perfect mth power. In § 3 sets of conditions that f_{3m} should involve a repeated factor are developed. Proof that the sets of conditions determined in the various cases really contain the minimum number is given in § 4.

§ 2. *Conditions that a p-ary Form may be an mth Power.*

Let

$$(1) \quad f_{3m} = x_2^m \phi_{0\,x_1/x_2} + x_2^{m-1}\phi_{1\,x_1/x_2}x_3 + x_2^{m-2}\phi_{2\,x_1/x_2}x_3^2 + \cdots + \phi_{m\,x_1/x_2}x_3^m,$$

where

$$x_2^{m-r}\phi_{r\,x_1/x_2} \equiv a_{m-r\,0}x_1^{m-r} + a_{m-r-1\,1}x_1^{m-r-1}x_2$$
$$+ a_{m-r-2\,2}x_1^{m-r-2}x_2^2 + \cdots + a_{0\,m-r}x_2^{m-r},$$

be a ternary form equal to the product of m linear factors. Then the (x_1, x_2) terms of those factors must be furnished by the linear factors of the binary form $x_2^m \phi_{0\,x_1/x_2}$. Let the factors of the latter be assumed to be known and let them be

$$x_1 + r_i x_2 \qquad (i = 1, 2, \cdots, t),$$

* Brioschi, "Sulla condizioni per la decomposizione di una forma cubica ternaria, etc." *Annali di Matematica*, ser. 12, vol. 7.

† Brill, "Ueber die Zerfällung der Ternärformen in lineare Factoren," *Jahresbericht der Deutschen Math.-Vereinigung*, 1896, and *Math. Annalen*, vol. 50.

‡ Gordan, " Das Zerfallen der Kurven in gerade Linien," loc. cit., vol. 45.

where $x_1 + r_i x_2$ is of multiplicity α_i.   We have proved in another paper * the following

THEOREM : *The ternary form f_{3m} can be factored into factors of the respective orders α_1, α_2, $\cdots$, α_t, which are rational and integral in the coefficients of the form f_{3m} itself on the one hand, and in the quantities r_i on the other, linear in the coefficients ; according to the formula*

$$(2) \quad f_{3m} = \prod_{i=1}^{t} \left[\frac{\partial^{\alpha_i}\phi_{0-r_i}}{\partial r_i^{\alpha_i}} x^{\alpha_i} - \alpha_i \frac{\partial^{\alpha_i-1}\phi_{1-r_i}}{\partial r_i^{\alpha_i-1}} x^{\alpha_i-1}y_i + \alpha_i(\alpha_i - 1) \right.$$

$$\left. \times \frac{\partial^{\alpha_i-2}\phi_{2-r_i}}{\partial r_i^{\alpha_i-2}} x^{\alpha_i-2}y_i^2 - \cdots + (-1)^{\alpha_i} \lfloor \alpha_i \phi_{\alpha_i-r_i} y_i^{\alpha_i} \right]$$

$$x_1 + r_i x_2 / x_3 = x / y_i \qquad (i = 1, 2, \cdots, t).$$

If the Hessian covariant of one of these factors of f_{3m}, considered as a binary form in (x, y), vanishes identically, f_{3m} has a linear factor of multiplicity α_i, and conversely.   Thus if $m = 2$,

$$f_{32} = ax_1^2 + 2hx_1x_2 + bx_2^2 + 2gx_1x_3 + 2fx_2x_3 + cx_3^2,$$

we get in this manner necessary and sufficient conditions that the *general* ternary quadratic be a perfect square.   These are †

$$C_1 = af^2 - 2fgh + bg^2 = 0,$$

$$C_2 = ab - h^2 = 0, \quad C_3 = ac - g^2 = 0.$$

They are independent, that is, as we shall show in general in § 4, they form a minimum set of conditions that a *general* form f_{32} be a perfect square.

We proceed now to generalize the results in the last paragraph, deriving necessary and sufficient conditions that the factorable p-ary form f_{pm} be a perfect mth power.

A notation can be adopted for f_{pm} which exhibits the form as a sum of $p - 2$ ternary forms plus a residue Ψ_{pm} which is not in the shape of a ternary form.   We use for the ternary summands a notation like (1), e. g.,

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* BULLETIN, vol. 17, No. 2, p. 63.
† Cf. Bromwich, Quadratic Forms, p. 9.

$$f_{pm} = x_2^m \phi_{0\,x_1/x_2}$$
$$+ x_2^{m-1}x_3\phi_{1\,x_1/x_2}^{(1)} + x_2^{m-2}x_3^2\phi_{2\,x}^{(1)}{}_{x_2} + \cdots + x_3^m\phi_{m\,x_1/x_2}^{(1)}$$
$$+ x_2^{m-1}x_4\phi_{1\,x_1/x_2}^{(2)} + x_2^{m-2}x_4^2\phi_{2\,x_1/x_2}^{(2)} + \cdots + x_4^m\phi_{m\,x_1/x_2}^{(2)}$$

(3)

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$+ x_2^{m-1}x_p\phi_{1\,x_1/x_2}^{(p-2)} + x_2^{m-2}x_p^2\phi_{2\,x_1/x_2}^{(p-2)} + \cdots + x_p^m\phi_{m\,x_1/x_2}^{(p-2)}$$
$$+ \Psi_{pm}.$$

When assumed to be linearly factorable, this equals

$$f_{pm} = \prod_{i=1}^{m} (r_{1i}x_1 + r_{2i}x_2 + r_{3i}x_3 + \cdots + r_{pi}x_p),$$

and if we take

$$a_{k_1 k_2 \ldots k_h \ldots k_p} = \sum r_{11}r_{12} \cdots r_{1k_1}r_{2k_1+1}r_{2k_1+2}$$
$$\cdots r_{2k_1+k_2} \cdots r_{hk_1+k_2+\ldots+k_h} \cdots r_{pk_1+k_2+\ldots+k_p},$$

we have

$$x_2^{m-k}\phi_{k\,x_1/x_2}^{(j)} \equiv a_{m-k0\overset{j-1}{\overbrace{0\ldots0}}k\overset{p-j-2}{\overbrace{0\ldots0}}}x_1^{m-k} + a_{m-k-110\overset{j-1}{\overbrace{\ldots0}}k\overset{p-j-2}{\overbrace{0\ldots0}}}x_1^{m-k-1}x_2 + \cdots$$
$$+ a_{m-k-hh\overset{j-1}{\overbrace{0\ldots0}}k\overset{p-j-2}{\overbrace{0\ldots0}}}x_1^{m-k-h}x_2^h + \cdots + a_{0m-k\overset{j-1}{\overbrace{0\ldots0}}k\overset{p-j-2}{\overbrace{0\ldots0}}}x_2^{m-k},$$

showing the notation for the coefficients of the form f_{pm}. We note that $\phi_{0\,x_1/x_2}^{(j)} \equiv \phi_{0\,x_1/x_2}$; and we take, without loss of generality, $a_{m00\ldots} = 1$.

Assume that the roots of $\phi_{0,\,-r}$ are all equal. Then $x_2\phi_{0\,x_1/x_2}$ is the mth power of a linear factor $x_1 + r_1x_2$ $(r_{21} = r_1)$. As is well known, a* necessary and sufficient condition for this is that the Hessian $H\phi_0$ of $x_2^m\phi_{0\,x_1/x_2}$ should vanish identically. By (2), when $H\phi_0 \equiv 0$, the typical ternary form in (3), $X^{(j)}$, say,

(4)
$$X^{(j)} = x_2^m\phi_{0\,x_1/x_2} + x_2^{m-1}x_{j+2}\phi_{1\,x_1/x_2}^{(j)}$$
$$+ x_2^{m-2}x_{j+2}^2\phi_{2\,x_1/x_2}^{(j)} + \cdots + x_{j+2}^m\phi_{m\,x_1/x_2}^{(j)},$$

is expressible in the form

$$X^{(j)} \equiv \frac{\partial^m \phi_{0\,-r_1}^{(j)}}{\partial r_1^m} x^m - m\frac{\partial^{m-1}\phi_{1\,-r_1}^{(j)}}{\partial r_1^{m-1}} x^{m-1}y$$

$$+ m(m-1)\frac{\partial^{m-2}\phi_{2\,-r_1}^{(j)}}{\partial r_1^{m-2}} x^{m-2}y^2 - \cdots + (-1)^m \lfloor m\,\phi_{m\,-r_1}^{(j)}y^m,$$

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*Other criteria may be had ; for instance, a necessary and sufficient condition that a binary n-ic f be an nth power is that the matrix of another binary n-ic ϕ with the bilinear invariant of f and ϕ should have an invariant factor different from unity.

where $x/y = (x_1 + r_1 x_2)/x_{j+2}$, a constant factor being neglected on the right.

In order that f_{pm} should be a perfect mth power, $x_2^m \phi_{0\, x_1/x_2}$ and the $p - 2$ binary forms $X^{(j)}$ $(j = 1, 2, \cdots, p - 2)$ must be mth powers. The vanishing of $H\phi_0$ and the $p - 2$ Hessian covariants of $X^{(j)}$ $(j = 1, 2, \cdots, p - 2)$ are necessary and sufficient conditions for this.

The Hessian of $X^{(j)}$ is

$$K_j = \begin{vmatrix} X^{(j)}_{xx} & X^{(j)}_{xy} \\ X^{(j)}_{yx} & X^{(j)}_{yy} \end{vmatrix},$$

and we have

$$X^{(j)}_{xx} = \lfloor m \sum_{h=0}^{m-2} (m - h)(m - h - 1) a_{m-h\overbrace{0\ldots 0}^{j}h\overbrace{0\ldots 0}^{p-j-2}} x^{m-h-2} y^h,$$

$$X^{(j)}_{xy} = \lfloor m \sum_{h=0}^{m-2} (m - h - 1)(h + 1) a_{m-h-1\overbrace{0\ldots 0}^{j}h+1\overbrace{0\ldots 0}^{p-j-2}} x^{m-h-2} y^h,$$

$$X^{(j)}_{yy} = \lfloor m \sum_{h=0}^{m-2} (h + 1)(h + 2) a_{m-h-2\overbrace{0\ldots 0}^{j}h+2\overbrace{0\ldots 0}^{p-j-2}} x^{m-h-2} y^h,$$

formulas which give explicitly, in terms of the coefficients of f_{pm}, the values of the determinants K_j.

Since the $p - 1$ binary forms $x_2^m \phi_{0\, x_1/x_2}$, $X^{(j)}(j = 1, 2, \cdots, p - 2)$ are all independent, the $p - 1$ Hessians are all independent. Thus *a set of necessary and sufficient conditions that a p-ary form f_{pm} which is linearly factorable, should be the mth power of a linear form consist in the simultaneous vanishing of the $p - 1$ binary Hessian covariants $H\phi_0$, $K_j(j = 1, 2, \cdots, p - 2)$.*

It will be proved in § 4 that when the $\binom{m+p-1}{p-1} - m(p - 1) - 1$ conditions that the *general* p-ary $m$ic be linearly factorable are joined to the above $p - 1$ Hessians we get a minimum set of conditions that f_{pm} (in general) be an mth power.

§ 3. *Conditions that f_{3m} Contain a Squared Factor.*

If we put $y = 1$, $m = 3$, $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in (2), carry out the indicated multiplications and divide by the coefficient of x^3, we get

$$(5) \quad f_{33} \equiv \prod_{i=1}^{3}\left[x_1 + r_i x_2 - \frac{\phi_{1-r_i}}{\phi'_{0-r_i}}\right] + \left(a_{10} - \Sigma \frac{\phi_{1-r_1}\phi_{1-r_2}}{\phi'_{0-r_1}\phi'_{0-r_2}}\right)x_1$$

$$+ \left(a_{01} - \Sigma \frac{\phi_{1-r_1}\phi_{1-r_2}}{\phi'_{0-r_1}\phi'_{0-r_2}}r_3\right)x_2 + \left(a_{00} + \frac{\phi_{1-r_1}\phi_{1-r_2}\phi_{1-r_3}}{\phi'_{0-r_1}\phi'_{0-r_2}\phi'_{0-r_3}}\right),$$

where the summations are to be taken with reference to the r_i, being symmetric in the r_i. Since f_{33} is factorable by hypothesis, the quantities in the parentheses are equal to zero. In fact these expressions equated to zero furnish a set of necessary and sufficient conditions that the *general* f_{33} be factorable into linear factors distinct term for term.

If we write

$$\Delta_1 = 3a_{30}\frac{\partial}{\partial a_{20}} + 2a_{21}\frac{\partial}{\partial a_{11}} + a_{12}\frac{\partial}{\partial a_{02}},$$

$$\Delta_2 = 3a_{03}\frac{\partial}{\partial a_{02}} + 2a_{12}\frac{\partial}{\partial a_{11}} + a_{21}\frac{\partial}{\partial a_{20}},$$

and let D be the discriminant of $x_2^3\phi_{0\ x_1/x_2}$ and R_3 the resultant of the latter taken with $x_2^2\phi_{1\ x_1/x_2}$, these three conditional relations become

$$H_3 = a_{00}D - R_3 = 0,$$

$$(6) \qquad H_2 = a_{01}D - \Delta_2 R_3 = 0,$$

$$H_1 = a_{10}D + \Delta_1 R_3 = 0.$$

If a ternary form f_{33} contains a squared factor it is, of course, linearly factorable as a whole. Then (2) gives

$$f_{33} = \left[x_1 + r_1 x_2 - \frac{\phi'_{1-r_1}}{\phi''_{0-r_1}}\right]^2\left[x_1 + r_2 x_2 - \frac{\phi'_{1-r_2}}{\phi'_{0-r_2}}\right]$$

$$+ \left(a_{10} - 2\frac{\phi'_{1-r_1}\phi_{1-r_2}}{\phi''_{0-r_1}\phi'_{0-r_2}} - \frac{\phi'^2_{1-r_1}}{\phi''^2_{0-r_1}}\right)x_1$$

$$+ \left(a_{01} - 2\frac{\phi'_{1-r_1}\phi_{1-r_2}}{\phi''_{0-r_1}\phi'_{0-r_2}}r_1 - \frac{\phi'^2_{1-r_1}}{\phi''^2_{0-r_1}}r_2\right)x_2 + \left(a_{00} - \frac{\phi'^2_{1-r_1}\phi_{1-r_2}}{\phi''^2_{0-r_1}\phi'_{0-r_2}}\right).$$

Evaluating the expressions in the parentheses we get a minimum set of conditions that the *general* f_{33} contain a squared factor, e. g.,

$$D = 0, \quad R_3 = 0,$$

$$\Xi_1 = a_{10}\frac{\partial D}{\partial a_{3-ii}} + \Delta_1\frac{\partial R_3}{\partial a_{3-ii}} = 0,$$

(7)
$$\Xi_2 = a_{01}\frac{\partial D}{\partial a_{3-ii}} - \Delta_2\frac{\partial R_3}{\partial a_{3-ii}} = 0,$$

$$\Xi_3 = a_{00}\frac{\partial D}{\partial a_{3-ii}} - \frac{\partial R_3}{\partial a_{3-ii}} = 0,$$

$$(i = \text{any number of the set } 0, 1, 2, 3).$$

In § 2 we have already developed a set of conditions that f_{33} be a perfect cube, assuming it to be linearly factorable. The method of this section gives us a minimum set, without assumption. For if the resultant and the first subresultant of the first partial derivatives of $x_2^3\phi_{0\,x_1/x_2}$ vanish, say $D = 0$, $D_1 = 0$, and also the resultant and the first subresultant of $x_2^3\phi_{0\,x_1/x_2}$ and $x_2^2\phi_{1\,x_1/x_2}$, say $R_3 = 0$, $S_3 = 0$, then (2) gives

$$f_{33} = \left[x_1 + r_1 x_2 - \frac{\phi''_{1-r_1}}{\phi'''_{0-r_1}} \right]^3 + \left(a_{10} - 3\frac{\phi''^2_{1-r_1}}{\phi'''^2_{0-r_1}} \right)x_1$$

$$+ \left(a_{01} - 3r_1\frac{\phi''^2_{1-r_1}}{\phi'''^2_{0-r_1}} \right)x_2 + \left(a_{00} + \frac{\phi''^3_{1-r_1}}{\phi'''^3_{0-r_1}} \right),$$

from which we get the necessary and sufficient conditions that the *general* ternary cubic be a cube, in the form

$$D = 0, \quad D_1 = 0, \quad R_3 = 0, \quad S_3 = 0,$$

$$\Gamma_1 = a_{10}\frac{\partial^2 D}{\partial a_{3-ii}^2} + \Delta_1\frac{\partial^2 R_3}{\partial a_{3-ii}^2} = 0,$$

(8)
$$\Gamma_2 = a_{01}\frac{\partial^2 D}{\partial a_{3-ii}^2} - \Delta_2\frac{\partial^2 R_3}{\partial a_{3-ii}^2} = 0,$$

$$\Gamma_3 = a_{00}\frac{\partial^2 D}{\partial a_{3-ii}^2} - \frac{\partial^2 R_3}{\partial a_{3-ii}^2} = 0$$

$$(i = \text{any number of the set } 0, 1, 2, 3).$$

We will now show still another method of applying the

theorem of § 2 in the determination of conditions for multiple factors of a ternary form f_{3m}.

Consider the case where f_{3m} has a single linear factor of multiplicity 2 and let it have, besides, $m - 2$ distinct linear factors. Then in (2) $\alpha_1 = 2$, $\alpha_2 = \alpha_3 = \cdots = 1$, and

$$\nabla_{r_1} = \phi'^2_{1-r_1} - 2\phi''_{0-r_1}\phi_{2-r_1} = 0.$$

Hence f_{3m} will have squared factors (and none of multiplicity higher than 2) when and only when there exist roots r_i common to the three equations

$$\phi_{0-r} = 0, \quad \phi'_{0-r} = 0, \quad \nabla_r = 0,$$

provided the resultant R of ϕ_{0-r} and ϕ''_{0-r} does not vanish. Thus if D is the discriminant of ϕ_{0-r}, and R_1 is the resultant of ∇_r and the greatest common divisor of ϕ_{0-r} and ϕ'_{0-r}, a set of necessary and sufficient conditions that a linearly factorable f_{3m} should contain linear factors to the second power is

$$(9) \qquad\qquad D = 0, \quad R \neq 0, \quad R_1 = 0.$$

§ 4. *Proof that the Conditions Give Minimum Sets.*

We can prove that the $\binom{m+p-1}{p-1} - m(p-1) - 1$ factorability conditions for any general f_{pm} together with the $p - 1$ Hessian conditions $H\phi_0 = 0$, $K_j = 0$ $(j = 1, 2, \cdots, p-2)$ of § 2 furnish a minimum set of invariant conditions in order that the *general* f_{pm} be an mth power.

The number of identical relations which exist among the elementary symmetric functions of m groups of p homogeneous variables is $\binom{m+p-1}{m} - 1$. If these variables are the mp coefficients of m linear factors of f_{pm}, the elementary symmetric functions are the coefficients of that form. If the factors are distinct, term for term, the identical relations, considered as relations among the coefficients of the form, will be $\binom{m+p-1}{m} - 1$ equations in $m(p - 1)$ non-homogeneous variables, so that only

$$\binom{m+p-1}{m} - m(p - 1) - 1$$

of the relations are independent. An independent set forms a minimum set of conditions that f_{pm} be factorable into m linear factors, distinct term for term. Now if f_{pm} is an mth power, the number of variables is only $p - 1$. Hence the minimum

number of conditions in order that f_{pm} may be an mth power
is $\binom{m+p-1}{m} - p$.

Each one of the Hessians $H\phi_0$, K_j $(j = 1, 2, \cdots, p - 2)$ is
of order $2m - 4$ in the variables which it contains, and so the
number of vanishing coefficients in each is $2m - 3$. Hence
these give $(2m - 3)(p - 1)$ conditions in addition to the
$\binom{m+p-1}{m} - m(p - 1) - 1$ assumed ones. But of the $2m - 3$
conditions obtained by equating to zero the coefficients of a
binary Hessian covariant only $m - 1$ are independent, as the
m coefficients of the form can all be expressed in terms of a
single quantity when the Hessian vanishes. Hence we have as
a total number of conditions given by the original factorability
conditions of f_{pm} and the Hessians

$$\binom{m+p-1}{m} - m(p - 1) - 1 + (m - 1)(p - 1) = \binom{m+p-1}{m} - p,$$

which is thus the minimum number required. Hence the rela-
tions derived in § 2 furnish a minimum set.

In the same way it may be shown that (6), (7), (8), (9) are
all minimum sets.

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THE GENERAL TERM OF A RECURRING SERIES.

BY PROFESSOR ARTHUR RANUM.

1. The principal theorem of this note expresses the general
term of a recurring series rationally in terms of the first few
terms and the constants of the scale of relation. Although I
derived it in 1908, I have only recently learned that practically
the same theorem was published by D'Ocagne in 1894 (*Journal
de L'Ecole Polytechnique*, volume 64, pages 151–224) and by
Netto in 1895 (*Monatshefte für Mathematik und Physik*, volume
6, pages 285–290). Nevertheless it may be worth while to
publish my own work for three reasons : first, because my
proof is simpler than those of D'Ocagne and Netto ; second,
because I have stated the result in a more explicit form than
that of either of these authors * ; third, because I have applied

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* D'Ocagne gives an explicit statement of the theorem (p. 163) for the
special case in which the series is a "suite fondamentale," but not for the
general case.