common the points α, β where L cuts V_3^2. With the conditions imposed it is clear that D must go through α (or β) and E through β (or α). . . . Suppose then that D goes through α and E through β. D and E are to be determined by the conditions that they shall be tangent to V_3^2 and such that the plane (DE) shall meet C_a, C_b, C_c respectively in one point. Consider now a fixed point a on C_a, and a point b on C_b. There are two V_3^1 going through a, b, α and tangent to V_3^2, and if c is the point other than α where one of them meets C_c, there are two V_3^1 tangent to V_3^2 and going through a, c, β. If b' is the point where one of them cuts C_b, it is seen at once that (b, b') are in (4, 4) correspondence, and for any of the 8 coincidences it is evident that we have two hyperplanes D, E which together with (A, B, C) form a system of the kind required.

It may be remarked in passing that

$$\prod_{i=1}^{i=4} x_i \sum_{k=1}^{k=4} \frac{a_k}{x_k} + x_5^3 = 0$$

represents a V_3^3 with four nodes of the second species, and is a mere generalization of the cubic surface with three such nodes represented by

$$x_1x_2x_3 + x_4^3 = 0.$$

We reserve for a later occasion the consideration of the special cases that may arise in the construction given above.

LINCOLN, NEB.,
January 22, 1912.

WHAT IS MATHEMATICS?

Principia Mathematica. By ALFRED NORTH WHITEHEAD and BERTRAND RUSSELL. Vol. I. Cambridge, 1910. Royal 8vo. xvi + 666 pp. $8.00.

THE game of chess has always fascinated mathematicians, and there is reason to suppose that the possession of great powers of playing that game is in many features very much like the possession of great mathematical ability. There are the different pieces to learn, the pawns, the knights, the bishops, the castles, and the queen and king. The board possesses certain possible combinations of squares, as in rows, diagonals, etc. The pieces are subject to certain rules by which their motions are governed, and there are other rules governing the players. A treatise on chess contains all these. Further however it also contains openings which have been found to be advantageous to one or the other of the players and usually contains also various endings of games for the tyro to analyse, in order that he may see how to acquire skill in foreseeing the situations that may arise in any game. One has only to increase the number of pieces, to enlarge the field of the board, and to produce new rules which are to govern either the pieces or the player, to have a pretty good idea of what mathematics consists.

In mathematics the game is much more complicated. The pieces we handle are the members of ranges of a more or less elaborate character. These members may be numbers, functions, lines, operations, any set of things we can define.* The moves on the board are groups of operations that may be performed upon these ranges and their members. We also must take into account a feature which is present in the game of chess in one move only—that of castling. In mathematics we may handle whole combinations of elements and operations upon them, as if they were single things. That is, we must take into account complexes of operations and ranges.

With these elements we do different things, according to our taste and ability. First of all there are the developments of *structure*. These include the construction of magic squares, and other questions of tactic, arrangements and combinatory analysis, factoring, decomposition of fractions, congruences, residues, and theory of form, through the structure of groups, up to finite fields, multiple algebra, calculus of operations, symbolic logic and general algebra. Then there are developments of the *invariants* that occur in different structures. We study algebraic and arithmetic forms, group characters, projective geometry, differential forms, topology, geometry in general, and operational invariants. (We mention necessarily only a few sample cases of the problems referred to.) There is also the study of correspondences of various types, the whole field of analysis or study of *functions*. In this are such things as the functions of a real variable, trigonometric series, algebraic functions, general analysis, geometrical transformations,

* Cf. E. H. Moore, "Introduction to a form of General Analysis," New Haven Math. Colloquium, 1910, page 2.

automorphic functions, calculus of variations, functions of a complex variable, vector fields, differential geometry, and rational mechanics. Further, and most difficult, are the studies in *inversion.* It is here that the expansions of the mathematical game take place. In this line of investigation we find algebraic corpora, ideals, modular systems, differential equations, integral equations, functional equations, and inverses of all kinds. Any theory of mathematics that would be complete is forced to account for all these different studies. To consider, for example, that one has laid the foundations of mathematics when he has produced the irrational number, is to confuse the theory of real variables with mathematics. Important as the problems of the continuum may be, the continuum is not the basis or foundation of structure. A knowledge of the different grades of ensembles does not enable one to ascertain whether a group is compound or not, or whether a number is prime or not. It does not determine the list of invariants of the decic, nor does it develop the differential parameters of a differential form of the fifth degree. The continuum has little to do with the properties of automorphic functions as functions. Cardinal and ordinal ranges do not play a prominent part in modular systems, nor in algebraic ideals, nor in functional equations. Neither likewise does a set of postulates for geometry, or some type of geometry, assist in determining how many associative algebras there are, built on twenty-four units. Indeed the postulates for associative algebras in general do not do this. So it becomes evident that when one wishes to discuss the principles of mathematics he must state what it is he refers to. If the analysis above (which was only indicated in a broad way) is correct, he may discuss the *ranges* with which mathematics has to deal, or he may discuss *operations* in general, or the principles may be those at the base of *multiple algebra.* He may mean the principles of mathematical composition and *form*, or the principles of the *invariance* in the transformations of forms, or the principles of *functionality* and correspondence in general, or the principles upon which may be founded the theory of *inversions.* For example, Russell's Principles of Mathematics was unable to handle the problem of the introduction of the imaginary into mathematics, and endeavored to crowd the whole theory of hypercomplex numbers into the theory of dimensionality. Peano's Formulaire did better, for the imaginary is defined at least by one of its examples. But the imaginary and the quaternion, and all other associative hypercomplex numbers, not to speak of those not associative, receive scant recognition in either case. A Principia Mathematica should cover the field, or it ceases to justify its title.*

Further we must not confuse mathematics and mathematical reasoning. It is true we infer in mathematics. But we also infer in physics, and history, and in daily life. Mathematics has no copyright on the process. To define mathematics as the science that draws necessary conclusions,† or as the class of all formal implications,‡ does not define at all. Other branches of human learning draw necessary conclusions, and formal implication is not unknown to them. Mathematics also uses the constructive imagination, the generalizing power, the intuition, and other mental processes. To define mathematics as "the study of ideal constructions (often applicable to real objects) and the discovery thereby of relations between the parts of these constructions, before unknown," is better.§ This definition brings out the *ideal*, the *construction*, and the *discovery*, three features which are essential to mathematical development. For example, an abstract group is ideal. Functions of the roots of an equation are ideal constructions. The discovery lies in seeing that the properties of the one will explain the relations involved in the other. Again, algebraic numbers are ideal. We construct with them domains of rationality, and arrive at the Galois theory in the relations.

However attention should be drawn to the fact that among the ideal elements with which we deal are many we invent or create entirely new. Examples are easily found. Quaternions were the result of Hamilton's attempt to extend the number field. The non-euclidean geometries were the attempt to create a new geometry. If any one fact stands out prominently in mathematical investigation it is this fact of the creation of new realms of investigation. Whether these ever are applied to real objects is a matter of less importance

* Cf. Poincaré, *Rev. Mét. et Morale*, vol. 13 (1905), pp. 815–835; vol. 14 (1906), pp. 17–34, 294–317, 866–868. Hilbert, Verh. III. Int. Math. Kongr. Heidelberg (1904), pp. 174–185. Russell, *Rev. Mét. et Morale*, vol. 14 (1906), pp. 627–650. Bôcher, BULLETIN, vol. 11 (1904), pp. 115–135.

† B. Peirce, *Amer. Jour. Math.*, vol. 4 (1881), p. 97.

‡ B. Russell, Principles of Mathematics, p. 3.

§ Century Dictionary (C. S. Peirce).

mathematically. It is a simple affair to invent even a new logic and mode of inference. Thus, let us imagine that the contradictive process were of period three, in place of two. That is, to the proposition p there is a first contradictory p', whose first contradictory is p'', the contradictory of the last being again p. What becomes of the conventional logic now? Yet by symbolism we can develop this kind of logic as well as any. This process we may call the mathematicising of logic. Indeed the volume before us really does something of this kind for logic, as the doctrine of types is close to the example above, and the result is labeled mathematical logic.

The outcome is interesting to mathematicians for several reasons. First of all it is a very general or abstract branch of mathematics. Secondly this book whose first volume is under consideration takes the place of a second volume of the Principles, in which the attempt was made to reduce all mathematics to symbolic logic, or as it is now called logistic. It is expected to demonstrate formally from these notions the derivation of the properties of cardinal and ordinal integers, irrationals, series, and eventually geometry and dynamics. We desire to examine the book from a purely mathematical point of view as to the success of the attempt.

There is an Introduction of three chapters. In some cases the fuller development farther on must be read in order to see exactly what the explanations of the Introduction mean. The first chapter gives a preliminary explanation of the notations used. These symbols are able to be themselves the elements of the entire development, and are thus very fundamental. A complete table is given here for the convenience of readers.

$\vdash$ it is true that..

$*$ chapter

$\sim$ contradictory of..

$\vee$ ..or..

$.$ ..and..

$\supset$ ..implies that..

$\supset_x$ ..implies that for x, formal implication

$=$ ..is identical with..

$= .\ Df$ nominal definition

$\equiv$ .. is equivalent to..

$\equiv_x$ .. is equivalent to.. for x, formal equivalence

P_Δ P-selections

$1 \rightarrow 1$ one-one correspondence

$\dot{-}$ negative of relation ()

$\subset\!\cdot$ relation () contained in..

$|$ relative product of..

$R‘$ the referent as to relation () of

$R“\,\alpha$ the referents of class α

$\breve{R}$ converse relation of R

$\dot{\exists}!$ there is a case of relation

$\dot{\Lambda}$ null-relation

$\dot{V}$ universal relation

$\upharpoonleft$ relation with domain limited to a class

$\upharpoonright$ relation with converse domain limited to a class

$\upharpoonright\!\!\upharpoonleft$ relation with field limited to a class

$1 \rightarrow$ Cls one-many correspondence
Cls name for classes
Rel name for relations
Cnv converse
Ex existent
Cls² class of classes
D‘ domain of relation..
Ɑ converse domain of..
C‘ field of..
$\rightarrow$, *sg*, referents of..
$\leftarrow$, *gs*, relata of..
x ♀ operator of x on ()
♀ y operator of () into y
α ♀ y values of x ♀ y for x over range α
Rl subrelation
Rl ex existent subrelation
φa proposition about a
φx proposition about a variable
$\varphi\hat{x}$ propositional function
$\varphi!\hat{x}$ predicative function
$(x)\cdot\varphi x$ proposition is true for all individuals x
$(\exists x)\cdot\varphi x$ proposition is true for some individuals x
$(\iota x)\varphi x$ the x with property φ
$\hat{x}(\varphi x)$ class defined by φ
Type range of x such that φx is significant.
$\dot{\cap}$ relation () and relation ()
$\dot{\cup}$ relation () or relation ()

$\downarrow$ couplet relation, vid.
$\cap$ common subclass
$\cup$ common superclass
$-$ negative of a class
$\subset$ contained in...
V universal class
Λ null class
$\exists$! there is a member of..
E! the member of .. exists
1 the class of unit classes
$1(\alpha)$ the class of unit classes of type α
φ, ψ, χ, θ, functional signs
$\rotatebox{180}{\iota}$ the...
ϵ is a member of the class determined by..
ι class of one member, unit class.
ι‘α the only member of α
Greek capitals, constants
Small Greek letters, usually classes
Capital italics, variable relations
p, q, r propositions
f, g functions
t‘x type in which x is contained
t_0‘α type in which α is contained
p‘κ product of classes
s‘κ sum of classes
Cl subclass
Cl ex existent subclass
‘ of

These symbols may be viewed in two different ways. They may be looked upon as furnishing a system of short-hand, or pasigraphy and stenography combined, intelligible to the initiated, and not only abbreviating the writing, but furnishing a mode of expression in which the usual color, shading of meaning, and associations of words are missing. This in itself would justify their use. Or we may look upon them as being symbols for the abstract elements of reasoning which have been found by the analysis, and for which no appropriate name exists. The latter formal view would not be taken by those who dislike to think that mathematics is the theory of certain combinations of symbols. But it seems to us that if we take the formal point of view, we are doing no more than when we define a rational fraction as a couple of numbers, subject to certain rules, and are able then to identify integers with those couples whose second number is 1. The foundations of arithmetic become solid. So too here if we consider that these symbols themselves, as representative of certain well-defined terms, are under consideration, we shall find a gain in clearness. In fact obscurity arises easily if we try to interpret some of the statements of the book in other ways.

We will undertake to give some notion of the ground covered in this first volume. All we can do of course in our limits of space is to discuss some of the prominent features of the book. The first thing we must consider is the meaning attached to certain words that are not used in the usual sense. The one we encounter at the beginning is the term *implication*. To this we need to devote a careful study, for it is the real basis of the further development.

Implication.

The startling statement is made early that "Newton was a man" and "The sun is hot" are equivalent propositions. From the definition of implication we see also that "Newton was not a man" implies that "The sun is hot," and "The sun is cold" implies "Newton was a man," or $2 + 2 = 4$, or "John Smith killed Pocahontas." This is a very different thing from what most of us would naturally call equivalence or implication. Implication, as used here, is a relation between two elementary propositions, or statements about particular individuals. We might raise the question as to whether there are any such propositions after all. But accepting for the time the assertion that they exist, implication is merely the statement that either the first proposition is false or else the second one is true. Now in the highly special sense in which we find terms used throughout the book, we feel instinctively that there is a certain artificial quality about every definition given or term used. Thus while we find the terms *true, false,* and *not true,* used, as well as *truth-value,* we find no real explanation given of the meaning of these words. Indeed we find later that there is a varying truth dependent upon the order of the statement. We must conclude then that the text throughout is concerned with a certain quality of the propositions considered, and not with the propositions themselves. When the authors talk about p and q they do not mean to discuss the significance of p and q but only a certain quality of p or of q. If the argument is about "Newton was a man," any other true proposition would do as well, for example, $2 + 2 = 4$. The assertions are not about the content of the propositions in either case but about the quality attached to either called its "truth-value." All propositions with the same truth-value are equivalent. They may in any implication be substituted, one for another. This explains how it is that "Newton was a man" and "The sun is hot" are equivalent. It also shows that the first implies the second, for either the first, irrespective of its significance, has the truth-value falsehood, or else the second has the truth-value truth. It is explained later (page 120) that we may throughout substitute the number 1 for any proposition with the truth-value truth, and the number 0 for any proposition with the truth-value falsehood, and reduce all formulas to the arithmetic of 0 and 1. The notion of truth-value is due to Frege,* although something similar is to be found in Boole.†

With regard to this view, we might suggest that whether the symbols p and q are to be regarded as equivalent or not depends upon the relationship they possess to other things in the universe, as well as upon their own significance. It would seem better to have used a different term to designate what the authors have in mind. A specific symbolism will make the matter clear. Let us agree to mark every proposition either with ° or with ′. This property of being tagged we will call T, thus

T (Newton was a man) is ′, T (The sun is cold) is °.

We may then state that what is meant by implication is one of the alternatives

$$\mathrm{T}(p) \text{ is } {}^\circ \text{ and } \mathrm{T}(q) \text{ is } {}^\circ, \text{ or } \mathrm{T}(p) \text{ is } {}^\circ \text{ and } \mathrm{T}(q) \text{ is } {}', \text{ or } \mathrm{T}(p) \text{ is } {}'' \text{ and } \mathrm{T}(q) \text{ is } {}''.$$

It is to be observed that T(q) is tagged ′ if T(p) is ′, otherwise it may be either. The one case excluded is evidently: tag of p is ′ and tag of q is °. We may admit that this is a simple thing, but (while it may be an idiosyncrasy on our part) it does not seem to be elementary. The notion of two tags, and of the property of being tagged, are clearly involved, and unless we make the tagging a purely haphazard affair, there is also involved the problem of determining *which tag must be placed on a given symbol.* The basal assumption (not mentioned) of the entire book seems to be, that we are in a position to say with regard to any proposition p about some specific thing whether it is true or not. Often it is the truth that we are

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* Russell, "The theory of implication," *Amer. Jour. Math.*, vol. 28 (1906), p. 160.

† Laws of Thought, 1854, p. 70.

endeavoring to discover. In fact, we have here the first example of the assertion that some parts of the book become more clear if we treat the results as purely symbolic, the whole being a calculus of symbols. If we know how to tag q there is no use in mentioning p. If p happens to be false, or if p happens to be true, we still confront the fact that we must decide that we have to choose between exactly the alternatives stated above. Hence for practical inference, this kind of implication seems to us to be worthless, and we therefore think another name would be desirable. The definition of implication, or rather the meaning given the implication sign, by Peano,* seems to be more fundamental, and elementary. It reads thus:

$$p)q$$

de p on déduit q; si p, alors q; la p a pour conséquence la q; la q est une conséquence de la p; la p est une condition suffisante de la q; la q est une condition nécessaire de la p.
We deal here directly with the propositions p and q and not with any functions of qualities they may possess. From the formal point of view, however, we have obtained a two-valued function of the indefinite set of marks, p, q, r, etc., that is, T( ) is $'$ or $^\circ$.

The propositional form that has a variable argument is generally expressed by φx, and we come next to the implications corresponding, that is, to *formal implication*. According to the Principles of Mathematics, the formal implication is the main thing in mathematics. It means that in φx we substitute for x any symbol (later it is restricted to a given type). The propositions resulting will each have a tag, as $'$ or $^\circ$, that is, as true or false.† The same is done with ψx, where ψ is a form into which we put the variable x in order to arrive at a proposition. Each of these propositions is tagged. Then φx implies (formally) ψx if in each case the tag on φx is $^\circ$ or else the tag on ψx is $'$, the same x occurring in each of the two. In other words, we must be able to assign for any given x the proper tag for φx and for ψx, and if we make out a three column table in which the first column is marked x, the second φx, the third ψx, and then enter the values of x as arguments, and the tags as values

* Formulaire, 4 éd., page 4; 5 éd., p. 3.
† If absurd, they are not propositions.

in the proper columns, we will necessarily have in the second column both ° and ′, and in the third column also both ° and ′, but with ′ in the second will always come ′ in the third. We might also state it thus: ψx must be true at least whenever φx is true. It is explained that we do not need to know or to produce every x about which the proposition may be stated. The φ and ψ are taken intensively. This is the first case we note in which there seems to be a lack of agreement between the theory and the practice. The definition calls for a comparison of the tags on two sets of expressions, which from their character would usually be infinite in number. As the direct comparison is impossible, the practical application goes back to a problem in intension, a term the authors endeavor to the utmost to shut out of the book. Again, if φx happens to be false for x in every case, we nevertheless have ψx implied. This may be true, that from false propositions anything may be concluded, but it does not advance mathematics very much. It would seem therefore that the attempt to found the whole system on the principle of truth-values (which we have called tags) is not so very successful, and that it would be better to make the undefined implication the base of the system. Indeed the authors apparently fall into this habit unconsciously. Thus we find as one of the assertions of the book

$$*2.04\ \vdash\ :.p.).q)r:):q.).p)r,$$

which they interpret: if r follows from q provided p is true, then r follows from p provided q is true. This reversal of conditions in the theory of functions would work havoc only too frequently. Of course the reading should be: consider {the tag on p is ° or else the tag on the statement [either the tag on q is ° or that on r is ′] is ′}, then if we mark all that has just been stated in { } with ′, we must mark also with ′ all that follows, viz. {the tag on q is ° or else the tag on the statement [either the tag on p is ° or else the tag on r is ′] is ′}. This is quite different from the Peano reading given just above.

We have dwelt upon the idea of implication as set forth here because this idea seems to be used more as a test of the accuracy of the results obtained than as a working notion. It is held in reserve as a court of last appeal. If one starts in directly with Section A and not with the Introduction, he does not encounter the notion of truth-value until *4.01 on page 120. If implication is taken as the fundamental notion and left undefined, we can define all the other symbols in terms of it and contradiction. This was done in the Principles, although the method used seems unnecessarily cumbrous. However, truth-value does not appear in the symbolism, and we have practically gained the following fundamental symbols:

φa, φb, etc., definite propositions about constant subjects a, b, etc.

φx, ψx, etc., definite propositional functions but variable arguments.

$)\cdots$, a relation between propositions, called implication.

Propositional Functions.

The propositional function is very important. It not only includes the usual predicate but may be any kind of a form with a blank place left for the entry of the argument. It is a symbol for the process that enables one to pass from a given argument term to another, the value term. The notation is as follows:

φa, a has the property φ, or of a we may say φ. This sentence is the value of φ for a.

φx, the propositional function applied to a variable argument. This is a symbol for any *one* of the values of the function, including statements which are not true as well as true statements.

$\varphi \hat{x}$, the function itself, as function. The $\hat{x}$ appears merely to assure the reader that the function really is a function of something or other. If the authors could have brought themselves to accept the Frege* notion and symbol $\varphi()$, the apparent argument could have been omitted.

$(x) \cdot \varphi x$, the entire list of values of φx are represented by this sign. In a large majority of them the truth-value would be $^\circ$ of course. Also the x is restricted to the range called the type of φ. The expression reads "φx is every case where x belongs to the type of φ."

$(\exists x) \cdot \varphi x$, there are values of x which give φx the truth-value $'$.

$\hat{x}(\varphi x)$, this symbol seems to have two meanings, at war with each other. In the early part of the book it is defined to mean the class (aggregate, ensemble) which consists of those arguments that make φx true. In other words, to be the set of

* See Principles, page 505.

individuals in the class determined by, or defined by, the function φ. Later it is identified with a symbol $\psi ! \hat{x}$ which is purely a function symbol, and does not represent individuals at all. This symbol practically defines the class property. The latter meaning seems to be the one which the authors expect to use, and may be interpreted to be the class as class, and not as individuals, but considered as a *denoting* symbol. This use of the symbol to represent the predicative function that would define the class collectively seems to be necessary in the system to enable us to use classes as arguments of functions. It is explained that we do not arrive at real classes thus, but only incomplete symbols. For example it is something like this. If we desire to say "The governors of states all wore silk hats," we must recast the statement to read "Certain persons were silk-hatted governors of states." This use of the class symbol $\hat{x}(\varphi x)$ and the function symbol $\varphi\hat{x}$ is close to that of the phrase "governor of a state," in two different senses, one meaning defining the qualifications necessary to be the governor of each of the states, the other defining the actual governors, so that they could be identified among other men. Both are functions. On this basis there are *no classes*, although the word class appears everywhere in the book. However, the claim is made that we have something just as good as a class, and in fact (page 84) "in mathematical reasoning, we can dismiss the whole apparatus of functions and think only of classes as "quasi-things," capable of immediate representation by a single name." An example would be the imaginary points of a curve.

However that may be as a matter of interpretation, we at least have arrived at two more symbols, from a mathematical point of view: the proposition as function, and the definition of solutions of a proposition. We have, in brief, isolated the function sign φ, and we can speak of "x such that φx." It would seem now that the fundamental thing after one has exhibited his set of elements with which he proposes to work, would be to consider functions of one variable, then synthetic processes by which these may be built up into useful structures. The importance of the propositional function is sufficiently insisted upon, but the uselessness of a mere set of isolated individuals is not dwelt upon. Nor is the character of synthetic processes in general examined at all. Of this we shall have something further to say.

Extensions.

Considerable stress is laid on the assertion that mathematics is primarily concerned with extensions, despite the fact that classes have become mere ghosts of themselves. An example would be any geometric theorem from Euclid. As "Let triangle ABC be isosceles, then triangle ABC has equal base angles." We are apparently discussing the single triangle ABC, but in reality we expect what we say to hold good for any triangle and thus for every triangle. We have the formal implication, which holds between the two sets of elementary propositions, one for each and every triangle. The ground of the reasoning seems to be an ambiguous case, and we seem to reason from any one to all. But is it so? If in a complex mental structure one chooses to pay attention only to certain features of the structure, and discuss them, ignoring the other accompanying features, are the statements about the whole complex structure, or about the portion abstracted? In proving the theorem cited above, does the color of the crayon used in drawing the triangle also enter the argument? Does the size of each individual angle and side also enter, or only certain relations they have? In finding the limit of the expression $2-(\frac{1}{2})^n$ do the particular values that n may take enter the argument at all? If one must answer no in these cases, that is to say, if one can abstract at all and reason about his abstraction, then we see no force in the constant appeal to extension. Is not the propositional function of the nature of an invariant rather? We say: *this* triangle has two equal sides, so *that* triangle, so also *yonder* triangle. In all the propositions of this sort that we choose to write down, we find the invariant phrase: two equal sides. It does not appear to be essentially different from any other invariant. If we were to conceive a transformation that could convert *this* into *that*, and *this* into *yonder*, the invariant of the transformation would in this case be: two equal sides, that is, isoscelism. Now the real question is, whether we can discuss invariants apart from the other circumstances in the concrete cases in which they are invariant. We certainly do this in mathematics. Indeed, is not this present book an attempt to discover what are the logical invariants in mathematics? In mathematics we are dealing with abstract elements all the time. The world of mathematics is an ideal, that is, an abstract world. We either abstract it from what we find in experience, or we create it de novo. For example, no such thing as *two* exists outside a mathematical mind, any more than the rainbow exists as color out there in the sky. We build our own structures and determine their relations, and no abstraction can be said to be more abstract than any other. In Socrates is mortal, Aristotle is mortal, Charlemagne is mortal, Socrates, Aristotle, and Charlemagne are abstract. No one would call them real men. They are as abstract as mortal, and as abstract as all men. Socrates is a word which represents the persistence of certain qualities of something day after day, and is thus in itself an invariant and therefore an abstraction. In one sense the whole of mathematics is the study of invariants.

If this analysis is correct, then when we say that the property φ implies the property ψ we do not need an extension to which to refer it. It is immaterial whether the extension is there or not. We can study the property as well in one case, as in a million, or an infinity of cases, if only we can isolate it itself. That is what we do in geometry, and in fact all through mathematical thinking. From this point of view (which may coincide with that which the authors denominate the philosophical, and with regard to which they admit the contention) mathematics is more concerned with intensions than extensions. These remarks apply to the extensional functions of functions. Indeed the paragraph in the middle of page 77 says:

". . . the functions of functions with which mathematics is specially concerned are extensional and . . . intensional functions of functions only occur where non-mathematical ideas are introduced, such as what somebody believes or affirms, or the emotions aroused by some fact."

It is difficult to see in this any more than the assertion that mathematics is not concerned with non-mathematical ideas, which no one would pretend to deny, unless it be some philosophers who thought that the Principles of Mathematics did not discuss mathematical ideas.

But there is a further point that we must notice. It is that the definition of function and class does not really produce the members of the extension at all. If we speak of the points of intersection of $x^2 + y^2 = 25$ and $x^2 + y^2 = 36$ what points are given as the extension of this proposition? If we speak of the roots of the equation

$$x^6 + 2x^5 - 13x^4 + x^3 - 7x^2 + 11x + 17 = 0,$$

where is the process in all the development of symbolic logic that will determine them for us? If we ask for the hypercomplex numbers that have the characteristic equation

$$\xi^4 - 4\xi\eta^2 + 5\eta^3 = 2$$

who in all the mathematical world will send us the list? Or lists, we should say, for this one equation will determine more than one certain set of such hypercomplex numbers. Quaternions satisfy the equation

$$q^2 - 2Sq \cdot q + T^2 q = 0,$$

but so do other hypercomplex numbers. Take the most definite function we can find, and what does it give? Not a class in extension, but certain properties that are found in some example we may produce, yet which may exist in an infinite number of other examples we have never thought of. Mathematics is full of discoveries of just such expansions of the extension of the notions that we have been using. It is a great advance, as Poincaré says, to find that we can bring two things, that is, two extensions under one name. As a simple case again, the notion of prime number is surely a definite thing, yet who knows how to ascertain whether 67989379012301 is a prime or not? The notion of simple group is definite, are there then simple groups of odd order, other than cyclic groups? When von Staudt called involutions on a line complex numbers, were they cases under the definition or not?

There is, as everyone knows, a vast difference between finding the value of a function for a given argument and finding the arguments that will satisfy a given function. In fact to meet the latter requirement mathematics has had to invent whole new extensions. Symbolic logic does not give us any assistance in this work of development nor any new methods. Its problem is only to criticise the character of the inferences involved in the process of development. It furnishes neither the major nor the minor premise but simply passes upon the validity of the transition from both to the conclusion. And all it furnishes in the propositional function is a sort of common invariant for many sets of classes. The classes may not exist in some senses, and may exist in others, just like the imaginary roots of an equation. But the propositional function does not point out the particular cases of members of the class it defines any more than it solves an algebraic equation. This is quite different from the explicit mathematical function,* like $\sin x$ or

$$\int_0^a \frac{\sin x dx}{x},$$

but analogous to the implicit function, like

$$\int \sqrt{y - x}\, dx, \text{ where } x^3y - xy^2 + y^5 = 4.$$

Types.

Chapter II of the Introduction develops the part of the Principia which differs most from the Principles, the *Doctrine of Types.* By this doctrine the authors hope they have resolved the paradoxes of the Principles as well as others that have been stated in discussions provoked by the original ones or by the theory of ensembles of Cantor.

The net result of the discussion seems to be that the predicate of a sentence is of the nature of a matrix or function symbol, and cannot serve as a subject for a sentence which has it also as predicate. Thus we might consider the statement "Triangle ABC is scalene." The functionality involved here, scalenity, cannot be put as the subject of this sentence. Scalenity is scalene, would be an absurdity. In this form at least, we might all admit that there is a sort of hierarchy of functions, if not of types. Of course one may talk of scalenity, but scalenity does not belong to the range that is itself scalene.

The *type* of a function is the class of objects that make it significant. That is to say in substituting values for x in φx some of them will give true propositions, some will give false propositions, some will give statements that are neither true nor false. That ensemble which produces with φ a proposition, true or false, is the type. The outcome is a little curious, as it leads to a reincarnated ghost of the buried *class.* The last paragraph of page 173 is interesting reading in connection with what has gone before. The difficulty seems to be of the same type as that which certain mathematicians find in recognizing any symbolic operator as an existent entity of the same character as the things it operates upon. Strictly speaking, of course there is a difference between $\frac{5}{1}$ and 5,

* Russell, *Revue Mét. et Morale*, vol. 13 (1905), pp. 906–917.

at least till we come to see that the invariant properties we spoke of above are the same in the one case as in the other, and we identify the two. In just this same way we elevate ensembles in general into the region of functions, and then we can draw a distinction between φ as function or φ as argument.

Whether we agree or disagree with the philosophy underlying the argument, we have gained one point more in our symbolism, that is, that we may make a function sign out of any symbol, simple or complex, and we may use any symbol as an argument for the proper function sign. In other words we may construct more and more complicated forms, and we may substitute for any single symbol or set of symbols, complex symbols. Incidentally we have a relativity theorem in mathematics in the doctrine of types. For no type is the bottom of all types. Types are only relative. An individual in today's discussion may be a function in tomorrow's. But do we resolve a contradiction by calling it absurd?

And it is difficult to see how the doctrine of types can be reconciled with many mathematical developments. Thus if we define the function φ by the differential equation $d\varphi x/dx = \varphi x$, we seem to have a case in opposition to the doctrine. For to define the derivative we must know an infinite set of values of the function φ, which is itself defined by means of the derivative. Apparently then we define a function in terms of the function itself. Another case more to the point possibly is the integral equation, and the integro-differential equation.

Relations and Descriptions.

The third chapter of the Introduction is devoted to descriptions, classes, and relations, under the title Incomplete Symbols. It is said that these can be defined only in their use. They are analogous to the symbols $\int$, ∇, $\sin^{-1}$, etc. However, it is pointed out that "the incomplete symbols are obedient to the same formal rules of identity as symbols which directly represent objects, so long as we consider the equivalence of the resulting variables (or constant) values of the propositional functions and not their identity. This consideration of the *identity* of propositions never enters into our formal reasoning." Under the limitations to the use of these symbols we find that, while x is always identical with x, yet the round square is not identical with the round square, for the reason there is no round square. This surely resembles the argument: Nothing is better than heaven, a shilling is better than nothing, therefore a shilling is better than heaven.

The descriptive symbol is $(\iota x)\varphi x$. It is used in precisely the same way as, and in fact differs little from, the symbol for a class. That is to say, it is a function sign, denoting the single object in the class. For example, the author of Waverly. This phrase does not point out the individual Scott, yet it identifies him. The legal John Doe does the same thing for a criminal who refuses to give his real name. These are, in the sense defined, function signs.

The relation is also a function, the function however having two arguments. There are many features common to these three incomplete symbols. The logic of relatives has also many developments that are not found in the others.

With regard to the three chapters of the Introduction, we desire to remark that they are in general somewhat difficult to apprehend as they now stand, for two reasons. First the distinctions drawn seem in many cases to be confused. It is sometimes difficult to ascertain whether the authors are using words in an every-day sense, in a philosophical sense, or in a purely technical sense. It is not always clear whether a symbol is under discussion, or the meaning of the symbol, or the use of the symbol. The introduction of many more good examples might have remedied this defect in style. In the second place, the statements do not seem to be thoroughly consistent. For example, it is not easy to decide what the authors mean by extensional. In the early part it seems to mean, as ordinarily in logic, the totality of individuals constituting an ensemble or collection. Later it seems to mean anything intellectual as distinguished from the emotional and the volitional. Then on page 196: "Propositions in which a function φ occurs may depend, for their truth-value, upon the particular function φ, or they may depend only upon the *extension* of φ. In the former case, we will call the proposition concerned an *intensional* function of φ; in the latter case an *extensional* function of φ." Also the apparent dread the authors exhibit towards the word *concept* seems to make the explanations often involved. The root of the whole difficulty seems to lie in an unconscious, or at least unstated, philosophical theory that a general notion is only symbolic and has no real existence of its own, but is existent only as it is manifested in some supposed concrete form. This notion seems to be the source of many of the peculiar interpretations forced upon the symbols. We may in most cases interpret them otherwise, as we have tried to point out; and as a calculus of logic, the system given here is very complete. We consider next the constructive features of the work.

Elements.

Substantially all that is meant by a *proposition* is to be found in the formal character of the two symbols p and $\sim p$. The latter is called the contradictory of p. It is of such a nature that the function $\sim$ is involutory, that is, $\sim\sim p$ is the same as p. If we understand then that we have a set of symbols p, q, r, or φa, φb, φc, and the like, with the duplicate set of their contradictories, we have the elements of the subject as analysed in this book. The descriptive symbol is substantially the same as we find exemplified mathematically in the symbol $+\sqrt{2}$, which in the present notation would be (ιx) $(0<x, x^2=2)$. That is to say, we introduce into arithmetic the indefinite symbol x with the agreement that we will insert it in our number series, and that for x^2 we will always write 2. This is, in a more general case, like the Kronecker modular theory. It is along the line of the algebraicising of mathematics, as opposed to the arithmetising of mathematics. It is immaterial from this point of view whether the thing x exists or not. If in any sense it does exist then we may use a single symbol for it rather than the long form (ιx) (φx). Practically the idea of class is of the same kind. A mathematical example is the definition of a class of algebraic numbers by an equation, as $x^4 + 5x^3 - 2x^2 + 3x - 13 = 0$. The idea of a symbolic class ought not to disturb a geometer, for he is used to imaginary points, circular points at infinity, and the like. We talk of these ideal things as if they existed, being careful that our phrases have meaning when we find the entities do exist in any sense. The same notion occurs again in the relative. A mathematical example is the equation of a curve. Thus $x^2 + y^2 = 25$ furnishes a relation, which finds as its proper representative the *curve* itself, while the pairs of arguments x, y furnish the points on the curve. The curve is a correlating agency for bringing together these pairs of arguments. Whenever there are entities which may be considered to be

represented by these symbols, then we use single letters for them and treat them in the old-fashioned manner of handling classes. This view of mathematics, and of propositions in general, we would prefer to call *functional,* rather then extensional, or intensional. It is surely mathematical, whether it is logical or not, and makes mathematics the fundamental basis of all reasoning, even more than the specialized interpretation called symbolic logic. On the philosophical side it seems to emphasize the statement that all reasoning in the last analysis is not about things but about relations between things, for every function expresses a relation between its argument and its value, and every relation may be referred to an ideal thing at least.

Combinations.

We must consider next the combinations that are actually built up out of these elements. The general development is given in Part I, which extends from page 89 to page 342, and is called Mathematical Logic. We desire to consider it apart from any meaning of a specific character that might be attached to the symbols. There is a single combination introduced, represented thus: $p \vee q$. Any two symbols may be joined in this manner. It is commutative, that is $p \vee q$ is the same as $q \vee p$. Special symbols are used for the combinations $\sim p \vee q$, which is written $p \supset q$, and the combination $\sim(\sim p \vee \sim q)$, which is written $p \cdot q$, the first being called implication, the latter logical product, while the basal combination is called disjunction. Any two of these symbols may be omitted. Indeed for ease of manipulation, it seems that to express everything in terms of the symbol $\bullet$ would be best. Thus for $p \vee q$ we write $\sim(\sim p \cdot \sim q)$, and for $\sim p \vee q$ we write $\sim(p \cdot \sim q)$. We agree further to certain permissible reductions or expansions. Thus, for example, we may write for p,

$$p \cdot p, \quad \text{or} \quad p \vee pq, \quad \text{or} \quad p \cdot q \vee p \cdot \sim q, \quad \text{or} \quad p \vee (p \cdot \sim p) \cdot q,$$

whatever q may be. In fact if we consider that in any product p and $\sim p$ are incompatible and such product may be dropped after the sign v, we arrive at one of the simple methods of handling this calculus. We agree further that if we have any expression φx, where x is variable, we may write the symbol $(\exists x)\varphi x$, and likewise if we have φx and φy(*9.1 and *9.11). Further we agree that in φx we may substitute for x any symbol a of the same type as x, and also that x may run over a given range of proper type (*9.14 and *9.13). Also we agree that in any expression φa, where a is a constant, we may consider φ by itself as a function, and vice versa (*9.15). We may now take any combination of symbols and build more complicated ones with the use of the v or the $\bullet$. We add, however, to the symbolism an expression for the range, thus: (x), meaning all values of x, and $(\exists x)$ meaning those values which are solutions. The name formal implication is given to the combination

$$(x)\varphi x)\psi x \text{ or } \varphi x)_x\psi x.$$

In using functions of two variables we introduce the abbreviations called relations. To indicate that x is a solution of φx we abbreviate thus: $x \epsilon \varphi\, !\, x$.

The formulas resulting from these few elements are very numerous, and no attempt will be made to go into them. It is from the results of these combinations however that the authors expect to produce other combinations which will have all the properties of numbers, series, etc.

It is obvious that, out of all the processes the mind goes through, others might have been selected as the fundamental ones. Whether a commutative combination is in the end more useful than one that is not commutative may be a question. We often must use a non-commutative product in mathematics, as "if p is *first* true, and then q is true, it follows r is true." The whole consideration of mathematical form* might from certain points of view be considered to be a prerequisite to the study of any kind of combination. In Whitehead's Universal Algebra this is partly in evidence.

We need to note further that the few modes of combination used here are really supplemented later by a free use of relational symbols, P, Q, R, ϵ, P_Δ, in fact by so many that when we remember the few combinations used here we wonder why these in particular should have been chosen to be represented by arbitrary signs. Indeed the number of arbitrary symbols which have to be memorized is so great in the book that one is willing to conclude that a more significant system could have been worked out. But taken as it is, there remains still the mathematical problem. Stated in brief it is this: given two operations by which from elements or marks new

* Cf. Kempe, *Trans. R. Soc. London*, vol. 177 (1886), p. 1–70; *Nature*, vol. 43 (1890), 156–162; *Proc. Lond. Math. Soc.*, vol. 26 (1894), 5–15.

elements or marks are produced, namely, let us say, given that from p, q we construct

$$s = \varphi p, \quad t = \psi(p, q), \quad \text{and} \quad u = \theta(p, q)$$

with the conditions or identities that

$$\varphi s = \varphi\varphi p = p, \quad \varphi\psi(p, q) = \theta(\varphi p, \varphi q) \quad \text{or} \quad \varphi\theta(p, q) = \psi(\varphi p, \varphi q),$$
$$\psi(p, q) = \psi(q, p), \quad \theta(p, q) = \theta(q, p).$$

We now have the properties of these combinations as combinations to consider. This problem is one of general algebra, universal algebra, or multiple algebra, according to the title preferred, and has at its base the very fundamental question as to what a combination is logically, psychologically, and otherwise; and what the operator φ, or θ, or ψ may be, what it does to the operand, what operators are derivable from it and how; further, what the result of the operation, say s, t, u above, is; how one may pass from the operand to the result, and reversely. These are elements that seem to be overlooked in the development as given in the book. Before one uses a calculus, in other words, he should investigate the laws of his calculus.

In the course of such investigation, it turns out that structural laws are very numerous. We may investigate laws that have been assigned purely arbitrarily, as for example those that actually have been so assigned in the study of multiple algebra. The reduction of all these divers arbitrary types of structure to a few simple forms is not possible, and the introduction of extra relational symbols merely furnishes a symbolism, but does not account for the forms nor does it show that they are deducible from the primitive forms laid down in this book as the basis of all reasoning. And if what is meant is that mathematical form consists of relations, then nothing is done beyond furnishing a mere name to a class of entities. Let us put it otherwise: to single out a few combinations as worthy of special signs, and to represent all others as relations, using letters, does not substantiate the claim that all terms have been defined in terms of the few combinations. The expressions for relations $\hat{x}\hat{y}\varphi(x, y)$ are combinations and the original $p \vee q$, $p \cdot q$, and $p)q$ can be so expressed; for each is not different from $\varphi(a, b)$ for a properly chosen φ, hence are cases of $\varphi(x, y)$, therefore define relations. To say that ultimately all logic is reducible to propositional functions, would then be the proper outcome. And logic becomes thus a branch of general algebra.

Prolegomena to Cardinal Arithmetic.

This constitutes Part II of the present book. Part III treating of Cardinal Arithmetic, Part IV of Ordinal Arithmetic, Part V of Series, are mentioned for the following volumes. The subjects treated in Part II are of high importance not only for cardinal arithmetic but for the ensemble theory. The divisions are Section A, Unit classes and couples; Section B, Sub-classes, Sub-relations, Relative types; Section C, One-many, One-one, Many-one relations; Section D, Selections; Section E, Inductive relations.

We find 1 defined here by the symbol

$$1 = \hat{\alpha}\,[(\exists x)\cdot \alpha = \iota`x] \quad \mathrm{Df.}$$

In words, 1 is the class of all unit classes, or since we have abolished classes per se, we will paraphrase this to read: 1 is a function satisfied by nothing but those functions which are true each in one case only. Or again, one is a property possessed by functions, namely, uniqueness of argument. For example, we speak of the author of Waverly, the President of the United States, the sin 30°, all these enable us to put the word *the* in evidence, and the *the-ness* in their character is that common property called 1. Whether this is a logical definition of the everyday 1 or not, what is accomplished is the construction of a symbol out of those already existing which defines the property of uniqueness. We agree to use 1 in place of the longer form $\hat{\alpha}[(\exists x)\cdot\alpha = \iota`x]$. Likewise 0 is defined to be the function satisfied only by those functions which are never true. For example, () is not identical with itself, () is true when its contradictory is true, and such like. These are in no case true, and the property of their impossibility is the number represented by 0. The cardinal 2 is defined similarly as a function satisfied only by functions which define couples. If the couple defined were an ordered couple, then we define 2_r, the ordinal 2. For example, 2 is a property possessed by quadratic equations, when we confine our attention to the solutions they have. From the purely symbolic point of view we use 0 and 2 to abbreviate the forms

$$0 = \iota`\Lambda \quad 2 = \hat{\alpha}[(\exists x, y)\cdot x \neq y \cdot \alpha = \iota`x \vee \iota`y] \quad \mathrm{Df.}$$

Mathematically we have thus found that if the words one, none, two are used, we must have in mind the *uniqueness* of certain classes, the *impossibility* of certain classes, or the *dyad character* of certain classes. We may accept or reject this view of what the symbols mean, but practically we have placed 0, 1, 2, in the list of symbols which form the range of classes of classes. They belong to the second order symbols. The symbolic point of view is nearly the same as saying that we start with objects, these are entities of any order m. Then we make a set of tags to enable us to distinguish the objects without being concerned with their other qualities. These tags are $m + 1$ order symbols (classes). We then make a set of symbols to enable us to talk about the tags. This set of symbols is the set of cardinal numbers, and is of the $m + 2$ order in the process of symbolizing or abstracting.

We begin now to reach arithmetic. As one example of what it looks like, we will quote the theorem which is to prove later that $1 + 1 = 2$. It runs thus

$$*54.43 \vdash : \cdot \alpha, \beta \epsilon 1 .) : \alpha \frown \beta = \Lambda \cdot \equiv \cdot \alpha \smile \beta \epsilon 2.$$

That is in English, if α and β are unit-classes with no common members, then their smallest superclass is a couplet.

In Section A we find also the ordinal 2_r, which does not differ from the class of alio-vids of C. S. Peirce; and also the ordinal $\dot{2}$, which does not differ from the class of vids. An ordinal 1 might be defined by $\dot{2} - 2_r$, which is the common subclass of vids and vids that are not alio-vids, that is the class of idem-vids. The connection with the cardinal 2 is in the fact that an alio-vid, that is, an asymmetric relation, must have two distinct terms. The relation here, being a vid, is between one object of thought and one other object of thought. Of course we are not far from the theory of quadrate algebras and matrices in general after we have arrived at this result.*

We pass over the next three sections, although they are of high importance, to Section E, which treats of generalized mathematical induction. The notion of *hereditary* class is defined, μ is a hereditary class with respect to the relation R if successors of μ's are μ's. For example if μ is the peerage, μ is hereditary with respect to the relation of father to eldest son. If μ is numbers greater than 100, μ is hereditary with respect to the relation of ν to $\nu + 1$. Mathematical induction is

* C. S. Peirce, *Amer. Jour. Math.*, vol. 4 (1881), p.221.

evidently included in this class of relations, and by means of it we pass from any finite integral case to any greater integral case, but finite. No such proof holds for any infinity. Thus we may say the binomial theorem is proved in this manner for *any* finite integral exponent, but not for *all* finite integral exponents. Indeed the word *all* here has no sense. We are led to consider powers of relations and the analysis of the field of a relation. This belongs to the general theory of operations.

The whole of the second Part really is mathematical logic of a little more specialized character than Part I, and this first volume could have properly been called a treatise on the mathematics of logic.

Summary.

In summary, the object that we have had in mind was to show that this first volume of the Principia is in reality an application of mathematical methods of definition and synthetic combination to the relationships between the abstract things that logic chooses to discuss. By means of a symbolism, which awkward as it is, is sufficiently comprehensive, a study is made of functions: as related to terms in propositions, and as shown in the particular forms of descriptions, classes, and relations. In one sense the highly ideal character of mathematical objects is made more evident. In another sense the real mathematical object, though already ideal, is sublimated still further into a logical object. The book is a culmination of the critical investigation of mathematical foundations of recent years, and will no doubt advance the systematization and mutual readjustment of mathematical treatments. It will assist in discovering tacit hypotheses, and in putting into formal shape the demonstration of many facts that have been brought to light by the intuition. If it eventually helps in any substantial manner to unify different theories and show their common features it will do enough.*

But while we may admit that it has perhaps placed the fundamental principles of the theory of ranges in a more definite form, and has done something for the theory of relations, we insist that the other great theories of mathematics are barely touched upon if, indeed, at all. These, we pointed

* M. Winter, *Revue Mét. et Morale*, vol. 15 (1907), pp. 186–216. E. Borel, ibid., pp. 273–383. H. Poincaré, ibid., vol. 17 (1909), pp. 451–482, 620–653. H. Dufumier, *Bull. des Sciences Math.*, vol. 35 (1911), pp. 213–221.

out, were the theory of structure and form, the theory of invariance, the theory of functions as functions, the theory of inversions. That these can receive a general treatment we do not doubt, inasmuch as some of them are receiving such development. In logistic then we find only a very definite *branch* of mathematics, and in this volume we have the most complete treatment of logistic that exists. The question that many have asked naturally "How far does it assist in building up synthetic systems of mathematics" is easily answered. It reaches arithmetic only after one volume of 666 pages. We would not expect the complete treatise then to furnish much that would be of a synthetic nature. Indeed that would be as unreasonable as to expect to build Eiffel towers and Eads bridges from a study of postulates and axioms for the foundation of geometry. While design rests upon these things in a sense, design antedates them just as language antedates grammar. It is not fair to the book or its aim to assert that it does nothing synthetic. Its problem is philosophical and analytical. It does enough if it shows us what are the characteristic features of reasoning and generalizes the types of reasoning. In this respect it is scientific as well as philosophical. It examines the rules of the great mathematical game. But it does not play the game nor undertake to teach its strategy.

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DIFFERENTIAL GEOMETRY.

Vorlesungen über Differentialgeometrie. Von LUIGI BIANCHI. Autorisierte Deutsche Übersetzung von MAX LUKAT. Zweite, vermehrte und verbesserte Auflage. Leipzig und Berlin, B. G. Teubner, 1910. xvi + 721 pp.

IN 1899 Guichard announced (*Comptes Rendus* 128, page 232) without proof the following theorems:

I. Let M be a point of a quadric of revolution Q whose axis is of length $2a$, F_1 and F_2 being the foci of Q and φ_1, φ_2 the points symmetric to F_1, F_2 with respect to the tangent plane to Q at M; let S be a surface applicable to Q; as S is applied to Q the points F_1, F_2, φ_1, φ_2 invariably fixed with respect to the corresponding tangent plane to Q take positions which