out, were the theory of structure and form, the theory of invariance, the theory of functions as functions, the theory of inversions. That these can receive a general treatment we do not doubt, inasmuch as some of them are receiving such development. In logistic then we find only a very definite *branch* of mathematics, and in this volume we have the most complete treatment of logistic that exists. The question that many have asked naturally "How far does it assist in building up synthetic systems of mathematics" is easily answered. It reaches arithmetic only after one volume of 666 pages. We would not expect the complete treatise then to furnish much that would be of a synthetic nature. Indeed that would be as unreasonable as to expect to build Eiffel towers and Eads bridges from a study of postulates and axioms for the foundation of geometry. While design rests upon these things in a sense, design antedates them just as language antedates grammar. It is not fair to the book or its aim to assert that it does nothing synthetic. Its problem is philosophical and analytical. It does enough if it shows us what are the characteristic features of reasoning and generalizes the types of reasoning. In this respect it is scientific as well as philosophical. It examines the rules of the great mathematical game. But it does not play the game nor undertake to teach its strategy.

JAMES BYRNIE SHAW.

DIFFERENTIAL GEOMETRY.

Vorlesungen über Differentialgeometrie. Von LUIGI BIANCHI. Autorisierte Deutsche Übersetzung von MAX LUKAT. Zweite, vermehrte und verbesserte Auflage. Leipzig und Berlin, B. G. Teubner, 1910. xvi + 721 pp.

IN 1899 Guichard announced (*Comptes Rendus* 128, page 232) without proof the following theorems:

I. Let M be a point of a quadric of revolution Q whose axis is of length $2a$, F_1 and F_2 being the foci of Q and φ_1, φ_2 the points symmetric to F_1, F_2 with respect to the tangent plane to Q at M; let S be a surface applicable to Q; as S is applied to Q the points F_1, F_2, φ_1, φ_2 invariably fixed with respect to the corresponding tangent plane to Q take positions which are denoted by F_1', F_2', φ_1', φ_2'; the loci of these respective points describe four surfaces for each of which the mean curvature is equal to $1/a$.

II. When the quadric Q is a paraboloid of revolution the loci of the points F_1' and φ_1' are minimal surfaces.

The announcement of these theorems marked the beginning of a new epoch in the theory of the deformation of surfaces. During the following years many eminent geometers attacked problems suggested by them, and in particular the deformation of the general quadric. This problem was solved in the admirable memoirs of Bianchi and Guichard which were jointly crowned by the French Academy in 1908. It is in the exposition of this theory and allied questions that the second edition of Bianchi-Lukat's Vorlesungen differs from the first, and is "vermehrt und verbessert." For the sake of brevity we shall refer to the two editions as (I) and (II).

Chapters 21 and 22 of (I) which deal respectively with N-dimensional space of constant curvature and hypersurfaces in spaces of constant curvature are omitted from (II), in order to make room for the theory of deformation of quadrics. It is to be regretted that this abridgment was necessary and that the treatise did not appear in two volumes so that this important theory, as well as topics subsequently referred to, could have been included. In view of this omission practically all of the quadratic differential forms which appear in the book involve only two variables and so there is no necessity of developing the theory for n variables as in (I). One finds accordingly that in (II) Chapter 2 is given over to the theory of quadratic differential forms in two variables. This makes the treatment simpler in places, but we cannot see that there was any gain made by the change. There is introduced an algebraic proof of the theorem: Two binary quadratic differential forms with the same constant curvature are always equivalent, and the transformation equations involve three arbitrary constants.

The other changes in the first part of the book are a few additions to the theory of deformation in general. Those double families of lines upon any surface S which become asymptotic lines on one of the deforms of S are called *virtual asymptotic lines*. The determination of these families of curves requires the solution of two simultaneous partial differential equations of the second order. Applying the theory of the Picard method of successive approximations, it is shown that any two intersecting analytic curves C and C' on an analytic surface S determine uniquely a double family of virtual asymptotic lines on S to which these curves belong, one to each family, and the deform of S on which these curves are asymptotic lines is analytic. This theory is applied in more detail to pseudospherical surfaces in Chapter 17, but it is similar to the treatment of this particular case given in Chapter 17 of (I).

At the end of Chapter 8 one finds a theorem due to Chieffi which is of great service in the subsequent discussion. If S is a non-ruled surface applicable to a ruled surface R, to the straight lines on R correspond a family of geodesics on S, called g. Let a be an asymptotic line of either system on S and draw the tangents to the curves g at points a, thus forming a ruled surface R_1. The theorem of Chieffi is that R_1 is applicable to S in such a way that a remains rigid during the deformation.

In Chapter 17 we begin to find traces of the results growing out of the theorems of Guichard. After developing the theory of Bäcklund transformations of pseudospherical surfaces as found in (I) (in places the treatment is slightly different), the author considers the real transformations obtained by combining conjugate imaginary ones, and more particularly the result of taking two opposite transformations B_σ and $B_{-\sigma}$ in applying the "theorem of permutability" (Vertauschbarkeitssatz). Let two such transformations be applied to a pseudospherical surface S, and let S_1 and S_2 be the resulting surfaces and S' the fourth surface which is the transform of these two; the normals to S_1 and S_2 meet in a point M_0 equidistant from these surfaces and likewise the normals to S and S' in a point M_0'. The two surfaces S_0 and S_0', the respective loci of M_0 and M_0', are applicable to the same surface of revolution and are complementary to one another. However, S_0 is not a pseudospherical surface as stated on page 483. According as σ is zero, real and different from zero, or pure imaginary, the meridian curve of the surface of revolution is different, when the latter surface is real. But for all three types of σ the surface S_0 is applicable to the surface of revolution of an imaginary ellipse. Thus one is brought incidentally into contact with the problem of deformation of quadrics.

With the exception of the first five pages the material in

Chapter 18 does not appear in (I). By means of a change of variables the equations of the preceding chapter are given a form which define a Bäcklund transformation of spherical surfaces, that is, surfaces applicable to the sphere, with the difference that all of the first transforms are imaginary. However, these may be combined into real transformations as described in the following theorem: If S_1 is an imaginary Bäcklund transform of a spherical surface S by means of a transformation B_{σ_1}, $(\sigma_1 = a + ib)$, and S_2 is the transform of S by $B_{\bar{\sigma}_1}$, $(\bar{\sigma}_1 = a - ib)$, the fourth surface S' given by the "theorem of permutability" is a real spherical surface. When $b = 0$, the normals to S and S' meet in a point M_0, and the normals to S_1 and S_2 in a point M_0'; the surfaces S_0 and S_0', the loci of these points, are complementary surfaces and are applicable to the same prolate ellipsoid of revolution. When $b = \frac{1}{2}\pi$, S_0 and S_0' are applicable to a hyperboloid of revolution of two sheets.

The second part of Chapter 18 is given over to the proof of the two theorems of Guichard stated at the beginning of this review, and converses of these theorems. In preparation for this discussion the author considers congruences of spheres, whose surface of centers S_0 undergoes a deformation, and also congruences of lines invariably fixed to a surface which is being deformed. If one of the sheets of the envelope of spheres has constant mean curvature as S_0 is deformed, the same is true of the other sheet. If two sheets are minimal surfaces, S_0 is applicable to a paraboloid of revolution P, the radius of a sphere being the distance between the corresponding point on P and the focus. If the mean curvature of the two sheets is different from zero and constant, S_0 is applicable to a prolate ellipsoid of revolution E or a two sheeted hyperboloid of revolution H, and the radius of a sphere is the distance between the corresponding point on E or H and a focus.

The Bäcklund transformations of a pseudospherical or a spherical surface S may be looked upon as transforming the surface elements of S, that is the points of S (called the *centers* of the elements) and small portions of the tangent planes about the points, into ∞^3 surface elements in space which can be grouped so as to be the surface elements of ∞^1 surfaces, the transforms of S. This point of view is not peculiar to these transformations. As a matter of fact it is thus that the author sets up the transformations of surfaces applicable to any quadric in the following very simple and remarkable manner: Let Q be any quadric and Q_1 a confocal quadric; let a surface element f have for center F, a point of Q, and for its plane π the tangent plane to Q at F; this plane π meets Q_1 in a curve C; take a point F_1 of C as the center of the surface element f_1 whose plane π_1 is determined by the line FF_1 and one of the rulings of Q_1 through F_1, it being understood that rulings of the same family are used in setting up the surface element for each point of Q; the plane π_1 is tangent to Q_1 at some point of the ruling. The fundamental theorem is the following:

THEOREM (A): *If Q is deformed in any manner into a surface S and in the deformation to each element f there are invariably coupled ∞^1 elements f_1, obtained by varying the point F_1 of C for each point F, these ∞^3 surface-elements can be correlated into a unique family of ∞^1 surfaces S_1, each of which is applicable to S and to Q.*

This transformation which converts S into a surface S_1 is called a transformation B_k, where the subscript k is a constant determining Q_1 among the quadrics confocal to Q. The proof of this theorem for the case when Q and Q_1 are hyperbolic paraboloids occupies, with certain allied problems, the whole of Chapter 19. The processes are in keeping with the general treatment of problems of differential geometry in the former chapters of the treatise and as a matter of fact it is essentially that followed by Bianchi in his prize memoir. The coordinates of Q and Q_1 are given in explicit form in terms of parameters referring to their generators. By straightforward steps the relations between the surface elements f and f_1 are established and it is shown that the correlation of the ∞^3 elements f_1, arising from the ∞^2 elements determined by a surface S, into ∞^1 surfaces S_1 requires the integration of an equation of the Riccati type. When the linear element of S_1 is calculated, it is found that it is not of the same form as that of S; consequently S and S_1 are not applicable in such a way that the centers of corresponding surface elements correspond in applicability, if at all. Since Q and Q_1 are confocal paraboloids of the same family, the orthogonal trajectories of the family establish a correspondence between Q and Q_1, namely corresponding points lie on the same trajectory. This is the transformation of Ivory of the one quadric into the other; it plays a very important role in this theory, as we shall see. Let S be any deform of Q and S_1 one of the surfaces defined by the correlation of the displaced surface elements f_1. If M_1 on S_1 corresponds to F_1 on Q_1, and $\overline{M}$ on S to the point F on Q which is the Ivory transform of F_1, the surfaces S and S_1 are applicable with the points $\overline{M}$ and M_1 in correspondence. The proof of this result covers ten pages most of which involves extensive calculation, so that at times the reader's interest lags, but throughout the whole treatment the geometrical ideas are extremely clear. There is no indication of the reason for hitting upon this correspondence of Ivory as the one leading to applicability. Later in the chapter certain properties of the Ivory transformation are derived as a means to the proof of the theorem: If S_1 is the transform of S by a B_k, then S is a transform of S_1 by the same B_k; these properties may have led to the suggestion of the trial of the transformation as a good guess, but the reader is left in the dark. The development of the results throughout the chapter is simplified and made more interesting by associating with the surfaces S the ruled surfaces R, applicable to S, as derived by means of the Chieffi theorem. Among these results one finds this pretty theorem: If the quadric Q rolls upon an applicable ruled surface R, each of the congruences Γ, $\overline{\Gamma}$ generated by the lines of the first and second system on a confocal quadric Q_1 may be arranged in a unique manner into a simple infinity of ruled surfaces R_1, each of which is applicable to Q and to R; moreover, the applicability of R_1 on R is accomplished by a continuous deformation. The surfaces R and R_1 of the foregoing theorem possess the following property in common with the non-ruled deforms of Q:

THEOREM (B): *The lines MM_1 joining corresponding points on a surface S and a surface S_1, resulting from S by a transformation B_k, constitute a W-congruence of which S and S_1 are the focal surfaces.*

The remainder of Chapter 19 is given over to the proof of Theorem (B) for the case where Q is a hyperbolic paraboloid. The method is much simplified by use of the results for the ruled surfaces R and R_1.

In Chapter 20 Theorems (A) and (B) are established for the case where Q and Q_1 are confocal hyperboloids of one sheet. The method followed is similar to that of the preceding chapter with very little abridgment. In fact the formulas differ only in form, and if anything they are more involved in the present case. Consequently the geometrical results are practically the same in the two chapters. From a historical point of view this fact is very interesting, since the earlier methods which were fruitful in the study of the deforms of paraboloids could not be applied in the case of central quadrics. It should be mentioned, however, that the exceptional case of hyperboloids of revolution leads to a theory of transformation of Bertrand curves analogous to the transformation of Razzaboni.

From the manner of construction of the surface elements f_1 it follows that if Q is real these elements are real only when Q_1 has negative curvature. Hence whatever be the type of Q, if only real, the surfaces S applicable to it admit real transforms S_1. However, the Ivory transformation makes real points on Q_1 correspond to imaginary points on Q, when the latter is not a hyperbolic paraboloid or a hyperboloid of one sheet. In this case the applicability of S and S_1 is *ideal.* These cases are treated in Chapter 21. By a slight change of notation the formulas of the two former chapters are used to establish Theorems (A) and (B) for the elliptic paraboloid, the hyperboloid of two sheets, and the ellipsoid. A novel converse result is that any real surface applicable in the ideal sense to a quadric of any of these three types admits of ∞^1 transforms S_1 applicable to the same quadric in the real sense. In a like manner there are real surfaces applicable in the ideal sense to the hyperbolic paraboloid and the hyperboloid of one sheet; such a surface is transformable by a B_k into a real surface applicable to the same quadric in the ideal sense. The discussion of these transformations closes with the following "theorem of permutability:"

If S is a surface applicable to a quadric Q, and S_1 and S_2 are two transforms of S by means of transformations B_{k_1} and B_{k_2}, where k_1 and k_2 are different constants, there is a fourth surface S', also applicable to Q, which arises from S_1 and S_2 by transformations B_{k_2}' and B_{k_1}', respectively.

For the proof of this theorem the reader is referred to the third volume of the second Italian edition. Here and in the memoir presented to the French Academy one will find also a treatment of surfaces applicable to the imaginary quadrics. One is inclined to feel that owing to the similarity of treatment of the several cases and in view of the existence of these details elsewhere much of Chapter 20 and part of Chapter 21 should have been omitted, and the space thus gained should have been used for other material appearing in the second Italian edition which sets forth new ideas. For example, there might have been included the treatment of surfaces with plane or spherical lines of curvature, or an exposition of the Weingarten method of deformation. If the latter had been developed in the manner of the second Italian edition, the ideas and formulas of moving axes would have been introduced, although rather late. We feel that the omission of this operator from the German editions is a serious one. It is to be greatly regretted that both the German and Italian second editions were not rewritten to such an extent that the moving axes could have been used to advantage.

The last three chapters deal with triply orthogonal systems of surfaces. The contents and treatment in the first two of these chapters are essentially the same as in the first edition. By far the greater part of the second chapter deals with confocal quadrics and geodesics on quadrics. This material could have been introduced into Chapter 6 and it would have served as a fine illustration of the general theory there set forth. We have always felt that this should have been done in the first editions. In fact we cannot see the reason for concealing this subject matter in a chapter which a beginner is not likely to read. The last chapter of (I) deals with Lamé families of pseudospherical surfaces. In the last chapter of (II) the treatment of spherical surfaces as well is included, together with that of certain surfaces applicable to quadrics.

We were pleased to find a fairly complete index in addition to a table of contents, which appears in (I) alone. The translation is well done, there are very few typographical errors and the bookmaking is up to the standard of the Teubners.

LUTHER PFAHLER EISENHART.