6. The curve of $S_4$ of which the double curve of $F'_{\mu\nu}$ is the projection is a complete intersection, but such is not the case for the double curve itself. In this respect the case where $\mu = \nu = 3$ is instructive. For then it is found that $t = 6\binom{3}{3} = 6$, $p_g = p_n = 5$,

$$\pi = 5\cdot 18 + 5 - \binom{8}{3} - 2\cdot 6 + 1 = 28,$$
$$h = \tfrac{1}{2}3\cdot 3\cdot 2\cdot 2 = 18 = 6\cdot 3 = 9\cdot 2.$$

The complete intersection of an $F_9$ and an $F_2$ with six triple points is of genus 63, while the intersection of an $F_6$ and an $F_3$ with six triple points is of genus 28, and yet the curve with the above characteristics cannot lie on an $F_3$, for then it would meet $F_9'$ in a curve of order 36. If $F_{\nu(\mu-1)(\nu-1)}$ is the intersection of $V_\nu$ and $V_{(\mu-1)(\nu-1)}$, then the curve in $S_4$ is on $F_{\nu(\mu-1)(\nu-1)}$, the residual intersection being of order $\mu\nu(\mu - 1)(\nu - 1)^2$, and its genus $p_1$ is found by remarking that if we project from another center this curve will be residual of one of order $\mu\nu(\mu - 1)(\nu - 1)$ and genus $\pi$, so that

$$\pi_1' = \pi + \tfrac{1}{2}\mu\nu(\mu - 1)(\nu - 1)(\nu - 2)[\mu\nu + \nu(\mu-1)(\nu - 1) - 4].$$

Thus, above, the residual curve is of order 72 and genus 397.

LINCOLN, NEB.,
*August* 20, 1912.

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## GEOMETRICAL OPTICS.

*The Principles and Methods of Geometrical Optics, especially as Applied to the Theory of Optical Instruments.* By JAMES P. C. SOUTHALL. New York, The Macmillan Company, 1910. xxiii+626 pp. with 170 figures.

THAT mathematicians and physicists have left the field of geometrical optics so largely to the scientific experts of the best firms of optical engineers may be but one of the signs of our ever increasing specialization and its accompanying narrowing of interests. Yet the association of such names as Euler, Fermat, Gauss, Hamilton, Kummer, Moebius, Sturm shows that once mathematicians contributed largely to the subject and were inspired by it; a similar state of affairs is true in regard to physicists.

From an impartial viewpoint it can hardly be gainsaid that there are at present more points of contact between geometrical optics and mathematics than between it and physics. For although too great a forgetfulness of the physical foundation (waves) as against the mathematical formulation (rays) may sometimes be inimical to obtaining correct results in geometrical optics,* the science of the physicist bears but lightly upon that of the optician and the efforts of the optician throw but little light upon those fundamental problems of physics which at present are the most absorbing.

To the mathematician, however, there is the opportunity of applying the theory of collineations in a manner which would make an interesting and instructive addition to any course on projective or descriptive geometry, there is a chance for the theory of congruences, especially normal congruences and focal surfaces, the calculus of variations may also find an application, and finally the Lie theory of contact transformations. Nevertheless it was not a pure geometer but Maxwell first, and Abbe later, who clearly perceived the seemingly obvious fact that the elementary theory of paraxial optical imagery was fundamentally a matter of collineations pure and simple, and it was Bruns, well-known for many a theoretical application of mathematics, who made such use of Lie's work in the more advanced parts of optical theory. It is indeed a pity that in all courses on contact transformations there should not be inserted enough of the applications to mechanics and optics to enliven the students' interest in the practical availability of the subject.

As Southall is more interested in this text in such parts of geometrical optical theory as lead to results applicable to the theory of optical instruments, he can give only minimal attention to the theory of congruences, the calculus of variations, and Lie's contact transformations, and he must content himself for the most part with mere references to other sources. He does, however, take up the collinear aspect in detail (pages 162–262, passim). It is to be prayed that, despite its practical tendencies, the text will appeal to mathematicians, and were it not for the hope that we may offer some encouragement to mathematicians to include in their courses at least a slight reference to optical applications, it is doubtful if we should have the incentive to compose this notice.

* Those who are now trying to revert somewhat to a corpuscular theory of light may succeed in replacing the wave theory by a ray theory, but such a theory would still have to account for such phenomena as at present fall under the topic of waves as distinguished from rays in the sense here used.

In the matter of book-writing on an intricate subject, the author has surely set us a model. The notation is most carefully chosen and consistently followed. To aid the reader there is an appendix of thirty pages in which the notations are tabulated with their appropriate explanations. This may at first seem too much of a good thing, yet it is a feature which cannot fail to be appreciated by students of geometrical optics. There are so many points—object points, image points, primary and secondary, focal points, "Flucht" points, etc.—which can best be kept distinct in a long treatise only by distinct notational conventions. There is also a necessary profusion of significant rays and planes, of angular and linear magnitudes, and of non-geometrical objects. The figures throughout the text are both numerous and well drawn; this alone is a considerable accomplishment and must have been a difficult task, but it is a great aid in rendering the clearly written text thoroughly transparent. The analytical index is long and critically constructed. The difficult proof reading must have been performed with exceptional care, but we have observed a few errors, all trivial.* References to other works are frequent though well selected, and the historical significance of the different advances in geometrical optics is not overlooked. In short the whole text manifests an affection of the author for his subject, and the care and industry which only such an affection could inspire.

It must be encouraging to the author to see the enthusiastic reception accorded to his work by various reviewing journals. The most flattering attention, however, which a work on geometrical optics could have is a translation into German; for the chief modern optical treatises and researches are German, and only a book of the highest excellence could receive an invitation into their society. Already a translation of Southall's Geometrical Optics is in preparation by Drs.

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* These are included (except for the peculiar arrow head on p. 391, line 9 from the bottom) in the list of seven furnished us by the author under the date of April 15, 1912.

P. 138. In Fig. 53 insert letter $N$ at end of dotted line $BC$.
P. 236. Formula at end of page should read: $Y = Z = \pm \sqrt{-f/e'}$.
P. 250. In formula (136), read $E_2'E' = -f_2e_2'/\Delta$.
P. 278. In the "legend" to Fig. 103, read $A_1F = E'A_2 = -1.5$.
P. 347. For "single spihercal surface" read "single spherical surface."
P. 558. In Fig. 167 insert letter $V$ at point of intersection of straight lines $MQ_1$ and $N_1D$.
P. 597. Instead of heading "$b$, $B$" read "$b$."

An additional list is given at the end of this review.

H. Schulz and Max Lange, scientific collaborators of the optical firm of Goerz. Let us hope that, despite the cost of the original production and of subsequent revisions, a generous publisher or some appreciative optical firm in this country will see to it that the necessary financial arrangements may be made to keep the revision of our American edition up to the German translation. It would further be highly desirable to furnish the author with every possible encouragement to bring out a second volume on the special theory of optical instruments, the natural continuation of the present volume.

The first two chapters of our text are introductory, descriptive of the methods and fundamental laws of geometrical optics and of characteristic properties of rays of light. Huygens's construction as modified by Fresnel is made the basis. Mention is made of that important principle whereby the solution of a problem in reflection can be derived from the solution of the corresponding problem in refraction by taking $-1$ for the relative index of refraction. The principles of least time and shortest route, Malus's law and the characteristics of an infinitely narrow bundle of optical rays (due to Sturm) are explained.

Chapters III and IV deal with reflection and refraction at a plane surface and refraction through a prism or system of prisms. In the discussion of the deviation of a ray obliquely refracted through a prism, there is repeated a widespread error leading to the erroneous formula (57) which should read

$$\sin \tfrac{1}{2}D = \sin \tfrac{1}{2}E \cos \eta_1, \quad \text{not} \quad \cos \tfrac{1}{2}D = \cos \tfrac{1}{2}E \cos \eta_1.$$

This error has apparently had a long and honorable career. It occurs in the works of Heath, Czapski, Loewe, Kayser, and practically all others. It was detected and pointed out by Uhler;* it had previously been pointed out by Larmor in 1898; it was avoided and the correct result was obtained by Mascart in his Traité d'Optique (1889). It is unlikely that any large number of authors working the problem through independently should have come upon the same erroneous formula; but of course it is impossible for each author to work out each detail independently; his very learning the subject from an earlier trustworthy source produces an aberration toward the error. The eradication of the newer errors of science is constantly

* *American Journal of Science*, volume 27 (1909), pages 223–228.

going on, but the older errors are hard to dislodge; the long survival and widespread adoption of such a one as this points a moral not to the latest author alone but to the whole scientific world.

Chapters V and VI discuss refraction and reflection of paraxial rays* at a spherical surface and through a thin lens or system of thin lenses. It is here that the applications of projective geometry begin. This is carried on to the general treatment of the geometrical theory of optical imagery in Chapter VII. The theory is applied in the next chapter to lenses and lens systems. Thus approximately one half the text is used up in dealing with the simple elementary geometrical optics of prisms and paraxial systems of rays refracted (or reflected) at spherical surfaces.

Now although the paraxial ray is simple to handle, it is not sufficiently effective for the optician; for the wide-angle bundle of rays is a necessity both for the distinctness and the brightness of the image. The ninth chapter therefore takes up the exact methods of tracing the path of a ray refracted at a spherical surface; spherical aberration is defined, and the refraction formulæ of Kerber and Seidel are given. Next follow the forms for calculating the path of a ray through a centered system of spherical surfaces. The discussion then is carried up to the general case of the refraction of an infinitely narrow bundle of rays through an optical system, and the matter of astigmatism which had previously been treated for refraction at a plane is now expounded for the case of a spherical surface.

The next, the twelfth chapter, on the theory of spherical aberrations, is by far the longest in the book, as may easily be imagined from the nature of its subject. The author follows chiefly the method of Seidel, the elegance of which is largely due to a selection of line coordinates felicitous for the rays involved. Gauss wrote the equation of the ray in the familiar form

$$y = \frac{Bx}{n} + P, \quad z = \frac{Cx}{n} + Q,$$

and used the parameters $B$, $C$, $P$, $Q$ ($n$ is the index of refraction and the $x$-axis is the optical axis). Seidel used as ray

* A paraxial ray is one lying so near the axis that magnitudes of the second order relative to its deviation from the axis may be neglected.

coordinates the quantities ($\eta$, $\zeta$, $\mathfrak{y}$, $\mathfrak{z}$) where ($\eta$, $\zeta$) and ($\mathfrak{y}$, $\mathfrak{z}$) are the coordinates of the two points in which the ray cuts two selected planes perpendicular to the $x$-axis. Especial attention is given to the terms of the 3d order, those of the 5th order being neglected.*

In Chapter XIII the question of color phenomena is treated, and the related subject of achromatism. The historical introduction and the description of what Jena glass has meant to optical engineering is an interesting relief from arrays of formulæ. The final chapter deals with the aperture and field of view, and the brightness of optical images. Here for the most part the mathematics is very simple. It is rare that the author applies physical reasoning instead of geometric, but a little is found in this last chapter and it recalls the fact that earlier he points out that Clausius from the second law of thermodynamics and Helmholtz from the law of conservation of energy could obtain the fundamental sine condition of Abbe.

It is clear that, from his consistent and logical presentation of the theory of geometrical optics, Professor Southall has attracted and must continue to attract highly favorable attention to himself, his institution, and his country. In this we should take pride. May he attract from us similar attention to his deserving subject.

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY.

EDWIN BIDWELL WILSON.

As the above review was about to go to press, the BULLETIN received from Professor Southall the following additional list of errata in his Geometrical Optics, supplied by R. S. Clay, Esq., of London. It is believed that the publication of the full list will be of material service to the readers of the book.

Page 159. Insert a minus sign before the first term on the right-hand side of formula (81).

Page 237. Change $x$ to $x'$ in the right-hand part of Fig. 92.

Page 304. In the first line of formulæ (191), read $(\alpha - \alpha')$ instead of $(\alpha + \alpha')$.

Page 306. In the legend to Fig. 122, read $\angle CBG = \alpha$ instead of $\angle GBC = \alpha$.

Page 307. In formula (197) read $\angle BCA_g$ and $\angle BCA_i$ instead of the symbols $\phi_g$ and $\phi_i$, respectively.

Page 310, near the bottom. Read $\angle BCA_g$ and $\angle BCA_i$ instead of $\phi_g$ and $\phi_i$, respectively.

* More recently in the *Astrophysical Journal* (May, 1911), the author has taken up the subject again.

Page 365. In formulæ (268) read $f_u = \bar{f}_u \cdot \cos^2 \alpha =$ etc.
Page 393. In the first term on the right-hand side of formula (298) in the denominator read $n_k - 1$ instead of $n_{k-1}$.
Page 414. In the second and third equations on this page, read $\Delta \frac{n \cdot \delta u}{uu}$ instead of $\Delta \frac{n \cdot \delta u}{u}$.
Page 414. In the fifth equation on this page, insert a minus sign before $d_{k-1}$.
Page 439. In the second term of formula (323), read $\bar{R}_m'$ instead of $R_m'$.
Page 461. In the third formula on this page, on the right-hand side, insert a multiplication-dot before the expression in brackets.
Page 462. At the top of this page, strike out the first line, and insert in place thereof the following:
If we neglect the terms of the 3d order, the direction-cosines of the incident ray may be regarded as 1, $b/n$, $c/n$; so that the approximate equations of the incident ray $BH$ are:
Page 462. In the denominator of the fraction on the right-hand side of the equation in the 6th line read $n^2(y_h^2 + z_h^2)$ instead of $y_h^2 + z_h^2$.
Page 465. In the last expression in the last term of the first of equations (355) read $\mathbf{J}_k^3$ instead of $\mathbf{J}_k$.
Page 465. In the second of equations (355) in the first term on the right-hand side read $(\mathbf{y}_1^2 + \mathbf{z}_1^2)\mathbf{z}_1$ in place of $(\mathbf{y}_1^2 + \mathbf{z}_1^2)\mathbf{y}_1$; and after the + sign before the last term insert ½.
Page 467. In the second of equations (357), in the numerator of the fraction in the first term on the right-hand side read $(\mathbf{y}_1^2 + \mathbf{z}_1^2)\mathbf{z}_1$ instead of $(\mathbf{y}_1^2 + \mathbf{z}_1^2)\mathbf{y}_1$.
Page 604. §122. Read Chap. I instead of Chap. II.

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## SHORTER NOTICES.

*A Treatise on the Analytic Geometry of Three Dimensions.* By GEORGE SALMON, late Provost of Trinity College, Dublin. Fifth edition, revised by R. A. P. ROGERS, Fellow of Trinity College, Dublin. Volume 1. London, Longmans, Green and Company, 1912. 8vo. xxii+470 pages, and two plates.

THE first edition of Salmon's Geometry of Three Dimensions was published in 1865. It formed the closing volume of an extensive treatise on algebraic geometry, two volumes of which were concerned with plane geometry, while the third contained a development and interpretation of the theory of linear transformations, from the standpoint of invariants, then just becoming known.

While many of the facts were known before, the point of view was a new one, and the great mass of material was