THE SPRING MEETING OF THE CHICAGO SECTION.

THE thirty-first regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, March 21–22, 1913, extending through four half-day sessions. The total attendance was eighty-one, including the following fifty-one members of the Society:

Professor R. P. Baker, Professor G. N. Bauer, Professor G. A. Bliss, Dr. R. L. Börger, Dr. E. W. Chittenden, Dr. G. R. Clements, Professor H. E. Cobb, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor E. W. Davis, Dr. W. W. Denton, Professor L. E. Dickson, Mr. C. R. Dines, Professor Arnold Dresden, Professor O. E. Glenn, Dr. T. H. Gronwall, Dr. T. H. Hildebrandt, Professor G. W. Hartwell, Mr. W. C. Krathwohl, Professor Kurt Laves, Professor A. C. Lunn, Dr. E. B. Lytle, Professor H. W. March, Professor W. D. MacMillan, Professor E. H. Moore, Professor F. R. Moulton, Mr. E. J. Moulton, Dr. Anna J. Pell, Professor Alexander Pell, Professor H. L. Rietz, Professor W. H. Roever, Dr. R. E. Root, Mr. A. R. Schweitzer, Miss Ida M. Schottenfels, Professor J. B. Shaw, Mr. T. M. Simpson, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Dr. E. B. Stouffer, Principal F. C. Touton, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor C. A. Waldo, Miss Mary E. Wells, Mr. C. W. Wester, Professor E. J. Wilczynski, Professor D. T. Wilson, Professor A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

Professor E. B. Van Vleck, President of the Society, presided at the two morning sessions and Professor D. R. Curtiss, Chairman of the Section, at the two afternoon sessions.

On Friday evening about fifty members dined together at the Quadrangle Club of the University, after which some attended a club lecture by Professor R. D. Salisbury, of the department of geography of the University of Chicago, while others spent the time in social intercourse in the club rooms.

At the business meeting on Friday afternoon the following officers of the Section were elected to hold office till the December meeting, 1913: Professor D. R. Curtiss, Chairman; Professor H. E. Slaught, Secretary; Professor A. L. Underhill, member of the program committee. The election of officers regularly occurs at the December meeting, but was not held in 1912 on account of the merging of the Section meeting in that of the Society at Cleveland. In connection with the election, the following resolution was unanimously adopted: That the Section favors the continuation of the custom of having a formal address from the Chairman upon his retirement from office. It was felt that there are too few papers giving a general survey of any field in mathematics and that this custom would insure at least a periodic opportunity of this kind.

Under the head of "informal notes and queries" several questions were briefly discussed, including a statement by Professor W. H. Roever with respect to the drawing of figures in space and the need by mathematicians of a better understanding of the principles involved. This need was acknowledged and the hope was expressed that we might have some adequate texts in English which could be used in outlining a course in this subject.

The following papers were presented at this meeting:

(1) Dr. T. H. Gronwall: "On various summation methods and their application to Fourier's series."

(2) Dr. R. E. Root: "Limits in terms of order, with example of limiting element not approachable by a sequence."

(3) Professor L. C. Karpinski: "John Caswell."

(4) Professor Karpinski: "The Whetstone of Witte."

(5) Professor R. P. Baker: "The topology of logical diagrams."

(6) Professor Baker: "The construction of the lines of a complex from given lines by ruler only."

(7) Professor W. O. Beal: "Concerning the stability of the eighth satellite of Jupiter."

(8) Professor G. A. Miller: "On the representation groups of given abstract groups."

(9) Miss Mildred L. Sanderson: "A fundamental theorem in the theory of modular invariants" (preliminary report).

(10) Mr. W. L. Miser: "On the solutions of linear homogeneous differential equations with elliptic function coefficients" (preliminary report).

(11) PROFESSOR R. D. CARMICHAEL: "On the Brownian movement and the ultimate constitution of matter."

(12) PROFESSOR CARMICHAEL: "On the impossibility of certain diophantine equations and systems of equations."

(13) PROFESSOR CARMICHAEL: "On certain diophantine equations having double parameter solutions."

(14) PROFESSOR F. R. MOULTON: "On orbits of ejection and collision in the problem of three bodies."

(15) PROFESSOR J. B. SHAW: "Formal determination of Clifford algebras."

(16) PROFESSOR E. J. WILCZYNSKI: "A new type of integral equations" (preliminary report).

(17) DR. E. W. CHITTENDEN and PROFESSOR A. D. PITCHER: "Classes which admit a development."

(18) DR. E. W. CHITTENDEN: "Relatively uniform convergence of series of functions."

(19) PROFESSOR G. N. BAUER and DR. H. L. SLOBIN: "Some transcendental curves and numbers."

(20) MR. C. E. LOVE: "Irregular integrals of the linear differential equation of the third order."

(21) MISS IDA M. SCHOTTENFELS: "A set of generators for quaternary linear groups."

(22) PROFESSOR G. A. BLISS: "A method of subdividing the area enclosed by a plane curve, with an application to Cauchy's theorem."

(23) DR. S. LEFSCHETZ: "The base for algebraic $(r-1)$-spreads immersed in an r-spread, with some applications."

(24) PROFESSOR E. L. DODD: "A justification of empirical probability based upon an unknown a priori probability."

(25) MR. K. P. WILLIAMS: "Concerning second order difference equations and the Schwarzian difference."

(26) PROFESSOR A. C. LUNN: "Integral equations in the kinetic theory of gases."

(27) PROFESSOR LUNN: "An integral functional equation in the theory of Brownian movements."

(28) PROFESSOR L. E. DICKSON: "Proof of the finiteness of modular covariants."

(29) PROFESSOR DICKSON: "On the rank of a symmetrical matrix."

(30) PROFESSOR DICKSON: "The projective geometry and covariant theory of a ternary quadratic form modulo 2."

(31) MR. J. MCDONALD: "On quadratic residues."

(32) Professor E. H. Moore: "On the geometry of linear homogeneous transformations of m variables" (preliminary communication).

(33) Professor O. D. Kellogg: "Conditions for a certain nomographic representation of a function of three variables."

(34) Professor O. E. Glenn: "Symbolic theory of finite expansions."

(35) Professor Kurt Laves: "Concerning a complete theory of the motion of the four minor satellites of Saturn."

(36) Professor Maxime Bôcher: "An application of Gibbs's phenomenon."

(37) Mr. A. R. Schweitzer: "On the working hypothesis in the genetic logic of mathematics. Second paper."

(38) Mr. A. R. Schweitzer: "A seeming contradiction in Poincaré's logical position."

(39) Dr. A. R. Crathorne: "The total variation of the isoperimetric problem with variable end points."

The papers of Miss Sanderson and Mr. MacDonald were communicated to the Society through Professor Dickson, and those of Professor Beal and Mr. Miser through Professor Moulton. In the absence of the authors the papers of Professors Karpinski, Miller, Carmichael, Dodd, Bôcher, Kellogg, Mr. Love, Dr. Lefschetz, Mr. Williams, Mr. MacDonald, and Dr. Crathorne were read by title. Abstracts of the papers follow in the order of the titles given above.

1. De la Vallée Poussin has defined the sum of any series $u_0 + u_1 + \cdots + u_n + \cdots$ as the limit (when it exists) for $n = \infty$ of the expression

$$V_n = u_0 + \sum_{v=1}^{n} \frac{n(n-1)\cdots(n-v+1)}{(n+1)(n+2)\cdots(n+v)} u_v,$$

and shows that when a function $f(x)$ has a generalized derivative of order k at a point x_0, then at this point the kth derivatives of the partial sums V_n of the Fourier series corresponding to $f(x)$ tend toward the kth generalized derivative of $f(x)$ for $n = \infty$ (*Bulletin de l'Académie Belgique des Sciences*, 1908).

In the present paper, Dr. Gronwall studies the relation between this summation method and that of Cesàro founded upon successive arithmetical means; the following propositions are established:

I. When a series $u_0 + u_1 + \cdots + u_n + \cdots$ is summable by Cesàro's means of any given order r, it is also summable by de la Vallée Poussin's process, with the same sum.

II. There exist series summable by de la Vallée Poussin's process, which are not summable by Cesàro's means of any finite order.

III. When $f(x)$ has a generalized derivative of order k at $x = x_0$, then at this point the kth derivatives of the Cesàro means of order $k + 1$ of the Fourier series corresponding to $f(x)$ converge toward the kth generalized derivative of $f(x)$.

Thus de la Vallée Poussin's theorem is implied by III (on account of I), but does not imply III (by reason of II).

2. While the conditions used in the paper by Dr. Root are fulfilled by any simply ordered class, provision is made for multiple interpretation of its postulates and theorems. The work pertains to a class of elements q of arbitrary character with a relation B of the type of "betweenness." The properties postulated for this relation are such as to persist under composition of systems. The relation B replaces betweenness in the definition of segment, and an element q is said to be a limiting element of any set that has an element distinct from q in every segment containing q. Certain fundamental theorems, including the proposition that every derived class is closed, result from the type of the system, without postulates. Very mild assumptions lead to a theory of multiple and iterated limits of functions, including as a special case the usual theorems on sequences of continuous functions. An additional postulate leads to a form of the Heine-Borel theorem, and to theorems on bounds of functions on a compact class. A sequence of elements is said to have the limit q if every segment containing q contains all but a finite number of elements of the sequence. On the basis of the postulates this definition of limit fulfils the conditions specified by Fréchet, thus securing, by means of the sequential definition of limiting element of class, a considerable body of theorems developed by Fréchet, Hedrick, and Hildebrandt. There arise, then, two parallel theories, of about equal extent, one based on the neighborhood definition of limiting element, the other on the sequential definition. That these two theories are not equivalent is shown by an example of a simply ordered class fulfilling all the postulates of the paper

and possessing a limiting element that is not the limit of any sequence of elements of the class.

3. Little has been known about the life of the English mathematician John Caswell (1655–1712), a contemporary of Wallis and the author of a somewhat notable work on trigonometry. Professor Karpinski has gathered the facts of his life largely from the publications of the Oxford Historical Society. Some light is thrown also on the general conditions with respect to the study of mathematics in the English universities during the seventeenth and early eighteenth centuries. The paper is to appear in the *Bibliotheca Mathematica.*

4. Some time ago the editor of the *Bibliotheca Mathematica* suggested the desirability of a study of Robert Recorde's "The Whetstone of Witte," with a view to determining the real contributions of this work to the development of algebra and algebraic symbolism. Professor Karpinski has made such a study. The net result would seem to be that the most noteworthy contribution of Recorde was the introduction of the equality sign. The works of Stifel, Scheybl, and Cardan were largely drawn upon by him. However Recorde contributed to the advance of the study of mathematics by furnishing in the English language excellent treatises from a pedagogical point of view.

5. In this paper Professor Baker shows that in a finite logical field two features are desirable, namely, integrity of the class regions and integrity of neighborhood. With line segments, as used by some writers, the first property is not available for $n > 2$. The second fails when areas are used when $n \geqq 4$. The first can always be obtained with areas. The plane representation is of lower genus than the Dyck representation of the corresponding abstract group. This reduction of genus occurs in other groups, which cannot be generated by two elements. The redundant generators used by Dyck in the case of infinite groups are not needed in finite groups, and they raise the genus. The minimum genus is given by a Cayley color diagram.

6. Plücker's construction of a complex, followed by many texts, uses the hyperboloid determined by three lines and its

intersections with one of the others. Cases exist in which all possible intersections are complex. To avoid this is desirable. The ruler construction which Professor Baker uses is based on incidences of lines and uses only planes of constructional lines.

7. Using the elements of the orbit of the eighth satellite of Jupiter given by Crommelin in volume 75, page 50 et seq., of the *Monthly Notices*, Professor Beal shows by applying the well-known criterion of stability which follows from Jacobi's integral, that the motion is stable, that is, that this satellite cannot become an independent asteroid. This is contrary to the conclusion reached by Kobb in the *Bulletin Astronomique* of 1908, page 414, from the first provisional elements that were given out from the Greenwich Observatory.

8. The abstract group G is said to be a representation group (Darstellungsgruppe) of the finite abstract group G', whenever G is one of the largest groups involving a quotient group which is simply isomorphic with G' as regards a subgroup M that is both in the central and also in the commutator subgroup of G. A number of fundamental theorems relating to representation groups were developed by I. Schur in two memoirs published in volumes 127 and 132 of *Crelle*. The present paper aims to establish some new results relating to these groups, and to derive some of the known theorems by simpler methods. Additional relations existing between the theory of representation groups and known theorems of abstract groups are also developed, especially as regards metabelian groups.

Among the theorems which have been established by Professor Miller are the following: Every possible group contains at least one commutator besides identity which is commutative with at least one of its elements. Every non-cyclic abelian group has at least two distinct representation groups, and there are infinite systems of representation groups of abelian groups whose commutator subgroups involve operators which are not commutators. If the square of an operator is commutative with another operator, their commutator is transformed into its inverse by the former of these two operators, and if a commutator is commutative with one of its two elements, its order is a divisor of the index of the lowest power to which this element must be raised to be commutative with the other element. The order of every representation group of a group of order p^m, p being a prime number, is of the form $p^{m'}$, and the orders of all of its commutators are divisors of p^{m-1}.

9. Miss Sanderson proves the following theorem: To any modular invariant i, of any system of forms under any group G of linear transformations with coefficients in the $GF(p^n)$, corresponds a function I of arbitrary variables formally invariant under G, and such that $I = i$ for all sets of values of the variables in the $GF(p^n)$. The proof rests upon the lemma: Let $a_1, \cdots, a_r$ be r arbitrary variables ($r > 1$), and $g_1, \cdots, g_r$ given elements of the $GF(p^n)$, $g_r \neq 0$. Then the determinants

$$N = \begin{vmatrix} a_1, & \cdots, & a_r \\ a_1^{p^n}, & \cdots, & a_r^{p^n} \\ \cdot & \cdot & \cdot \\ a_1^{p^{(r-1)n}}, & \cdots, & a_r^{p^{(r-1)n}} \end{vmatrix}, \quad D = \begin{vmatrix} a_1, & \cdots, & a^r \\ \cdot & \cdot & \cdot \\ a_1^{p^{(r-2)n}}, & \cdots, & a_r^{p^{(r-2)n}} \\ g_1, & \cdots, & g_r \end{vmatrix}$$

are such that N is divisible by D in the field and the quotient $Q = N/D$ has the properties:

$Q \neq 0$ if $a_1 = g_1, \cdots, a_r = g_r$; $Q = 0$ if $a_1 = e_1, \cdots, a_r = e_r$, where $e_1, \cdots, e_r$ are elements of the field not proportional to $g_1, \cdots, g_r$.

If $a_1, \cdots, a_r$ are the coefficients of the forms, a fundamental system of invariants which are formally invariant is formed out of sums as to $g_1, \cdots, g_r$ of powers of these Q's, where the $g_1, \cdots, g_r$ are the coefficients of a particular set of forms in the system.

10. In this paper Mr. Miser treats of the forms and properties of the multiple-valued solutions of the linear homogeneous differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \psi_{ij}(t) x_j \qquad (i = 1, \cdots, n),$$

where the $\psi_{ij}(t)$ are elliptic functions whose primitive periods are ω, ω', and whose poles are of order one. When the n roots of the indicial equation for each singular point are all distinct modulo unity, it is shown how single-valued doubly periodic functions $\phi_j(t)$ can be determined so that the general solution is

$$x_i = \sum_{j=1}^{n} A_j e^{\int \phi_j(t)dt} \xi_{ij}(t) \qquad (i = 1, \cdots, n),$$

where the ξ_{ij} are regular functions in the whole finite part of the plane.

11. In this paper Professor Carmichael proposes a new law of action among the ultimate mechanical units of matter. The law is suggested by the behavior of the Brownian particles as they are observed under the microscope, and especially under the ultra-microscope. As an analytical expression of the law we have the following: As the distance x between any two particles of matter varies, the force of attraction varies as $f(x)$, where

$$f(x) = \frac{1}{x^2} + \frac{a_4}{x^4} + \frac{a_6}{x^6} + \frac{a_8}{x^8} + \cdots,$$

$a_4, a_6, a_8, \cdots$ being properly determined constants. These constants are in general different for different kinds of matter. Corresponding to the fact that a particular kind of matter may in general exist in three and but three forms, namely, as a solid, a liquid, a gas, are certain relations among the constants $a_4, a_6, \cdots$. Other fundamental properties of matter also have their analytic correspondents; as, for instance, the fact that all gases have so nearly the same physical properties. The value of this law in the explanation of the processes of nature will depend on how well it lends itself to the use of the experimenter. Immediately it suggests numerous tests to be applied in the laboratory and it would thus appear to be an instrument of value.

12. The object of Professor Carmichael's second paper is to prove the impossibility of certain interesting diophantine equations and systems of equations. In two cases the argument is carried out by means of Fermat's famous method of "infinite descent." The paper will be offered for publication in the *American Mathematical Monthly*.

13. In his third paper Professor Carmichael obtains a double

parameter solution of each of several diophantine equations of which the following are typical:

$$x^3 + 2y^3 + 3z^3 = t^3, \quad x^3 + y^3 + z^3 = 2t^3,$$
$$x^3 + y^3 + z^3 + u^3 = 3t^3, \quad x^4 + y^4 + 4z^4 = t^4.$$

14. The problem under consideration by Professor Moulton is that in which an infinitesimal body moves subject to the attraction of two finite masses. Orbits of ejection and collision are those in which the infinitesimal body leaves one of the finite bodies and collides with the other one. The existence of infinitely many of these orbits is established. They are of importance in the consideration of the escape of molecules from the atmosphere of one body to another, and also in the consideration of periodic orbits, because certain types of these orbits are the limits of families of periodic orbits.

15. Professor Shaw's paper is a deduction of the properties of a Clifford algebra of N dimensions from the defining equation $\alpha\beta = -\beta\alpha + I\alpha\beta$. A Clifford algebra is one whose units consist of a fundamental set $i_1, \cdots, i_n$, their products two at a time, three at a time, etc., with the equations

$$i_s^2 = -1 \quad (s = 1, \cdots, n), \qquad i_s i_t = -i_t i_s \quad (s \neq t).$$

16. The integral equations considered by Professor Wilczynski are linear equations, both of the Volterra and the Fredholm form, and include equations of the first as well as of the second kind. The novelty consists in the form of the integral which is regarded, in the simplest case, as an open line integral independent of the path.

17. In terms of a general class P which admits a development Δ as defined by E. H. Moore, Drs. Pitcher and Chittenden have developed a theory analogous to that of Fréchet, in which the results obtained by Fréchet, Hedrick, Hildebrandt and others are secured through appropriate conditions on the development Δ of P.

18. Dr. Chittenden considers the immediate consequences of the definition of relatively uniform convergence of series of functions given by E. H. Moore, and shows that if the set

Q of points, at which the measure of convergence of a series of functions defined on an interval (a, b) is greater than zero, is reducible, the series converges relatively uniformly as to a scale which admits a definition in terms of the given series and set Q. He shows further that if for a series of continuous functions there is a perfect set on which the set Q is dense, then the series converges relatively uniformly as to no scale function. Osgood has given examples of such series.

19. In this paper Professor Bauer and Dr. Slobin trace some of the consequences of Lindemann's theorem that in the equation $e^x = y$, x and y cannot both be algebraic numbers (*Mathematische Annalen*, volume 20, page 224). Let $\varphi_1(x, y)$, $\varphi_2(x, y)$, $\varphi_3(x, y)$ be of the form $P_0y^n + P_1y^{n-1} + \cdots + P_n$, where the P's are polynomials in x, and where all the constants involved are algebraic numbers; further let there be no common factors in φ_1, φ_2, and φ_3. Then the following theorems are established: In the equation $\varphi_1(x, y) \cdot e^{\phi_2(x, y)} + \varphi_3(x, y) = 0$, x and y cannot both be algebraic numbers, except for a finite number of pairs of values determined by the simultaneous equations

$$\varphi_1(x, y) = 0, \qquad \varphi_2(x, y) = 0$$

and

$$\varphi_1(x, y) = 0, \qquad \varphi_1(x, y) + \varphi_3(x, y) = 0.$$

All the direct and inverse trigonometric and hyperbolic functions represent transcendental numbers for all algebraic values of the argument, except zero. Equations of the form

$$\varphi_1(x, y)e^{\phi_2(x, y)} + \varphi_3(x, y) = 0,$$

$$\varphi_1(x, y) + \varphi_2(x, y) \tanh \varphi_3(x, y) = 0,$$

$$\varphi_1(x, y) + t = 0,$$

where t is any transcendental number and where, instead of the hyperbolic tangent, any other direct or inverse circular or hyperbolic function may be substituted, represent families of curves which pass at most through a finite number of algebraic points.

By expanding $f_1(x)$, $e^{f_2(x)}$, $f_1(x) \log f_2(x)$, $f_1(x) \sin f_2(x)$, etc., where $f_1(x)$, $f_2(x)$ are explicit algebraic functions of x, series are obtained which represent transcendental numbers whenever an algebraic number is substituted for x.

20. Mr. Love's paper considers the existence and form of asymptotic solutions for the differential equation

$$y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = 0,$$

where the coefficients are developable, for large real values of x, in asymptotic (or convergent) series of the form

$$a_r(x) \sim x^{rk}\left(a_{r,0} + \frac{a_{r,1}}{x} + \cdots\right) \quad (r = 1, 2, 3).$$

Detailed study is made of the various cases in which the so-called characteristic equation has multiple roots, since these cases do not appear to have been treated heretofore. The corresponding problem for the equation of second order has been considered in an earlier paper of which the present work forms a continuation.

21. In volume 12 (1905) of the BULLETIN Miss Schottenfels published the paper, "A set of generators for ternary linear groups." In the present paper she develops a similar set of generators for quaternary linear groups of the type $x_i = x_{i+1}$, $x_k = x_1 + hx_2$ $(i = 1, 2, \cdots, k-1;\ h = 0, 1)$.* It is proved that all substitutions of the form

$$\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4,\\
x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4,\\
x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4,\\
x_4' &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4,
\end{aligned}$$

with coefficients rational integers of determinant unity, can be formed by combinations of the substitutions

$$T = (x_2 \cdot x_3 \cdot x_4 \cdot x_1), \quad S = (x_2 \cdot x_3 \cdot x_4 \cdot x_2 + x_1).$$

It is also proved that all matrices of the form

$$(\alpha_{ij}) = \begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{\ 4} \end{matrix} \qquad , j = 1, 2, 3, 4,)$$

* Moore, *Mathematische Annalen*, vol. 51, p. 436.
Schottenfels, *Annals of Mathematics*, ser. 2, vol. 1, no. 8. BULLETIN, VOL 6, pp. 440–443.
Burnside, *Messenger of Mathematics*, vol. 24, p. 109.

where α_{ij} are marks of the GF $[2^1]$ and (α_{ij}) is of determinant unity, can be formed by combinations of

$$T = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{matrix}, \quad S = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{matrix},$$

where T and S generate a group holoedrically isomorphic to the alternating group upon eight letters, $G_{8!/2}$.

22. Consider a simply closed continuous rectifiable curve which divides the plane into an exterior and an interior. Professor Bliss shows that the interior of such a curve can be divided by a finite number of segments of straight lines into regions whose diameters are less than ϵ, the number of auxiliary segments necessary being not greater than $4^{L\sqrt{2}/\epsilon}$. The theorem has a number of applications, one of which may be of special interest. With its help Cauchy's theorem in the theory of functions can be easily proved for any rectifiable curve if it has first been deduced for a rectangle. The methods of Goursat and Moore are readily applicable to a rectangle, but involve difficulties when applied to a curve which is supposed only to be rectifiable, on account of the complicated way in which a rectifiable curve may intersect a straight line.

23. In this paper Dr. Lefschetz gives the extension to an r-spread V_r of the theory of the minimum base developed by Severi and Poincaré for algebraic surfaces. Let $\{\Phi\}$ and $\{\Psi\}$ be two continuous systems, ∞^2 at least, of $(r-1)$-spreads immersed in V_r, $\{\varphi\}$ and $\{\psi\}$ the systems, ∞^2 at least, which they cut out on the general variety of a system $\{\Pi\}$. Then if φ and ψ are equivalent on Π when the latter varies, the same holds for Φ and Ψ, that is, there exists a continuous system in V_r containing them both totally. If ρ, ρ' and σ, σ' are the numbers of Picard and Severi for V_r and Π respectively, then $\rho \leqq \rho'$, $\sigma \leqq \sigma'$, from which follows the existence of a minimum base on V_r. Hence if, for any system such as $\{\Pi\}$, $\rho' = \sigma' = 1$, we have also $\rho = \sigma = 1$.

In particular if V_r has no singularities whatever, then all the $(r-1)$-spreads immersed in it are complete intersections — a known result. Also if V_r is the locus of ∞^1 $(r-1)$-flats then $\rho = 2$, $\sigma = 1$. Hence if ϕ_i is an $(r-1)$-spread in such a

V_r, a_i its order, α_i the order of its section by the flats, N the number of points common to r such spreads in V_r, then Story's generalized formula,

$$N = \Sigma a_1 \alpha_2 \cdots \alpha_r - (r-1)\mu \prod_1^r \alpha_i,$$

follows at once. Finally a new invariant number is introduced by considering the base cut out in the canonic system of V_r, and it is shown to exist even if V_r has no canonic system.

24. If an insurance company finds that, of 1,000 men of a given age, 990 are alive at the end of one year, what justifies the acceptance of .99 as the probability that a man of the given age will live at least a year? Dr. Dodd uses an unknown a priori probability in the following theorems. (1) Suppose that an event of unknown probability p has occurred just r times in s trials. Let the probability a priori that p lies in the interval $I = (x_1, x_1 + \delta)$ be $\int_{x_1}^{x_1+\delta} \psi(x)dx$; where $\psi(x)$ is limited and integrable; and, moreover, has a positive minimum in (0, 1), or at least in that portion of $(\sigma, 1 - \sigma)$ which remains when a finite number of sub-intervals such as $(\xi - \sigma, \xi + |\sigma)$ are removed, σ being taken small at pleasure. Then the probability a posteriori that p lies in I is greatest, with a chosen δ, when $x_1 = (r/s) + \eta$, where $\lim_{s=\infty,\, \delta=0} \eta = 0$, uniformly with respect to r. Roughly speaking: The most probable value of p is r/s when s is large enough. (2) If $\psi(x)$ satisfies the conditions specified in (1), the probable value of p will differ from r/s by less than any preassigned positive ϵ when s is large enough. (3) If $\psi(x)$ satisfies the conditions of (1), and s_1 additional (future) trials are to be made, then the most probable number of future occurrences is one (or possibly both—so to speak) of the two integers nearest to $(r/s)s_1$, provided that s, the number of past trials, is large enough.

25. Mr. Williams determines a difference equation of the third order which is satisfied by the quotient of any two solutions of a homogeneous linear second order difference equation. This third order equation has the property that, if any particular solution of it can be obtained, then the problem of solving the second order equation is reduced to finding a solution of a first order equation. An expression analogous

to the Schwarzian derivative of a function is obtained, and it is shown in a simple direct manner that this expression is invariant for a linear fractional transformation with periodic coefficients.

26. A fundamental problem of the kinetic theory of gases is the determination of the distribution of the molecules in space and velocity, by solution of Boltzmann's differential-integral equation for assigned physical conditions. For the case where the molecules are thought of as hard elastic spheres Hilbert has recently shown that, if the solution be sought in the form of a power series in a guiding parameter, each step of the computation corresponds to the solution of an integral equation of the Fredholm type with symmetric kernel.

The first part of Professor Lunn's paper gives another proof of this reduction, of a more geometric character, extending the theorem, moreover, so as to assume simply that the molecules have kinetic spherical symmetry, that the mutual forces during an encounter are such as to conserve momentum and energy, and never lead to union of two molecules, and that a certain condition of convergence is satisfied. In all cases the kernel is a function of the absolute values of two local molecular velocities and of the angle δ between them.

The second part of the paper shows that if the kernel be thought of as expanded in a series of Legendre polynomials in $\cos \delta$, and the known functions in the integral equations in series of surface harmonics in the velocity space, then the corresponding expansions of the unknown functions are determined by the solution of certain Fredholm equations in the one-dimensional realm of the absolute value of the velocity.

The first term of the power series was shown by Hilbert to be the well-known Maxwell formula. The general form of the second term is here found to contain, besides the additive solutions of the homogeneous equations, only harmonics of the first and second orders, corresponding respectively to thermal gradient and viscous stress, and leading to a computation of the coefficients of thermal conduction and viscosity. The value of the latter coefficient given by Meyer, and that of the former resulting from one of Tait's equations, which seems to be little known but fits experimental conditions better than Meyer's, both prove to be only the first terms in series resulting from solution of the integral equations by iteration.

27. In this paper Professor Lunn points out an obscurity in Einstein's treatment of the theory of Brownian movements, which consists in reducing the problem to the differential equation of diffusion. An alternative treatment is suggested, based on a kind of property of transitivity of the distribution function, leading to a non-linear integral functional equation, of which the Fourier-Einstein formula gives a special solution. Other special solutions are found, such as to show that the integral equation alone does not imply either the differential equation or certain auxiliary conditions suggested by the physical problem. The question is raised whether the solution is determinate when these other conditions are included.

28. In the first paper by Professor Dickson, it is shown that all rational integral modular covariants of a system of forms in n variables are rational integral functions of a finite number of such modular covariants. The proof makes use of the universal covariants* of the general linear modular group and of a lemma† which states that any set of monomial functions of n variables contains a finite number of functions $M_1, \cdots, M_f$, such that any function of the set is the product of some M_i by a monomial function. The paper will appear in the *Transactions*.

29. The second paper by Professor Dickson gives a short, elementary proof of Kronecker's theorem that in a symmetrical matrix of rank $r > 0$ at least one principal minor of order r is not zero.

30. In the third paper by Professor Dickson, certain invariants and covariants of the ternary form $F = b_1x_1^2 + \cdots + a_1x_2x_3 + \cdots$ modulo 2 (later shown to form a fundamental system) were obtained by a study of the (infinite) projective geometry in which the homogeneous coordinates x_1, x_2, x_3 of a point are roots of any congruences modulo 2 with integral coefficients. The polar of the point (y) with respect to F is given by a determinant whose rows are a_1, a_2, a_3; y_1, y_2, y_3; and z_1, z_2, z_3. The polar thus passes through (y) and the apex (a) of the conic $F \equiv 0$. Any line through (a) is a tangent to the conic, whose line equation is thus $\kappa = a_1u_1 + a_2u_2$

* *Transactions*, vol. 12 (1911), p. 75.
† *Amer. Journal of Math.*, October, 1913.

$+ a_3u_3 = 0$. Since the a's are cogredient with the x's, the function Δ obtained by replacing x_i by a_i in F is an invariant, the discriminant of F. In the polar form, take $(y) = (x)$, $z = (x^2)$. Then

$$K = \begin{vmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}$$

is a covariant of F. For the typical non-degenerate form $F = x_1x_2 + x_3^2$, we have $K = x_1x_2(x_1 + x_2)$, and the resulting three lines meet the conic in three points, which with the apex (0, 0, 1) define a complete quadrangle covariantly associated with the conic. The diagonal points of this quadrangle lie on the line $x_1 + x_2 + x_3 \equiv 0 \pmod{2}$. Applying a linear transformation which replaces the special form F by the general one having $\Delta = 1$, the special line yields the linear covariant

$$L = \Sigma(\beta_i + 1)x_i, \qquad \beta_1 = b_1 + a_2a_3 + a_2 + a_3, \cdots.$$

The invariant $\beta_1\beta_2\beta_3$ of L is an invariant of F. Forming the corresponding function for $F + \lambda\kappa^2$, we obtain an invariant of F and κ, from which we deduce a formal contravariant of the second order of F. L has the same relation to the latter that κ had to F.

31. The object of Mr. McDonald's paper is: (1) To give a direct proof of the reciprocity theorem in the case of the number 2 based on the binomial theorem; (2) To show how the quadratic character of an odd number m with respect to an odd prime n may be determined, from the binomial theorem, without the separation of m into its factors; (3) To exhibit the identity between a certain function of m and n and the Legendre-Jacobi symbol (m/n), when m and n are odd numbers.

32. Let a linear homogeneous transformation T of the m-space of m cartesian coordinates have the s distinct roots w_g of respective multiplicity m_g $(g = 1, \cdots, s)$; the determinant of the transformation $T - wI$ (I = identity) is $(w_1 - w)^{m_1} \cdots (w_s - w)^{m_s}$ with $m = m_1 + \cdots + m_s$. The following canonical form of the transformation T is well known:

$$T\varphi_{ghij} = \varphi_{ghi\,j-1} + w_g\varphi_{ghij},$$

where the φ_{ghij} are certain m linearly independent points or vectors of the m-space: $g = 1, \cdots, s$; $h = 1, \cdots, k_g$; $i = 1, \cdots, l_{gh}$; $j = 1, \cdots, n_{gh}$, with $n_{g1} > \cdots > n_{gk_g}$ and $l_{g1}n_{g1} + \cdots + l_{gk_g}n_{gk_g} = m_g$; further $\varphi_{ghi0} = 0$. The vectors φ_{ghij} form a basis of the m-space canonical for the transformation T. The positive integers k_g, l_{gh}, n_{gh} are the same for all canonical bases (φ_{ghij}).

In his note Professor Moore determines the linear $\overline{m}$-spaces of the m-space which are invariant as to choice of canonical basis, that is, spaces such that, if for a canonical basis (φ_{ghij}) the vector $\Sigma_{ghij}a_{ghij}\varphi_{ghij}$ belongs to the $\overline{m}$-space, so does the vector $\Sigma_{ghij}a_{ghij}\varphi'_{ghij}$ for every canonical basis (φ'_{ghij}). Such an $\overline{m}$-space is invariant under T, or, if T is singular, it at least transforms by T into a sub-space of itself. For the $\overline{m}$-space the roots w_g are of multiplicity $\overline{m}_g$ $(0 \leqq \overline{m}_g \leqq m_g)$ and a canonical basis $(\overline{\varphi}_{ghij})$ is a part or all of a canonical basis of the m-space, viz., $\overline{\varphi}_{ghij} = \varphi_{ghij}$ for $g = 1, \cdots, s$ with $\overline{m}_g \neq 0$, $h = 1, \cdots, k_g$; $i = 1, \cdots, l_{gh}$; $j = 1, \cdots, \overline{n}_{gh}$, where the numbers $\overline{n}_{gh}$ are subject to the conditions

$$0 \leqq \overline{n}_{gh} \leqq n_{gh};$$

$$\overline{n}_{g1} \geqq \cdots \geqq \overline{n}_{gk_g} \geqq 0;\ n_{g1} - \overline{n}_{g1} \geqq \cdots \geqq n_{gk_g} - \overline{n}_{gk_g} \geqq 0.$$

If $\overline{n}_{gh} = 0$, the corresponding vectors $\overline{\varphi}_{ghij}$, understood to be 0, are to be omitted from the canonical basis $(\overline{\varphi}_{ghij})$; if every $\overline{n}_{gh} = 0$, the $\overline{m}$-space is simply the origin $(\overline{m} = 0)$. Further the canonical basis $(\overline{\varphi}_{ghij})$ may need for each g a slight notational readjustment in view of the possible equality of some of the numbers $\overline{n}_{g1}, \cdots, \overline{n}_{gk_g}$.

The invariant $\overline{m}$-space characterized by the integers $\overline{n}_{gh}$ consists precisely of the vectors φ of the m-space which are of the form

$$\varphi = \sum_{gh}\varphi_{gh} \quad (g = 1, \cdots, s;\ h = 1, \cdots, k_g),$$

where for every gh the vector φ_{gh} satisfies the conditions

$$(1) \qquad (T - w_gI)^{\overline{n}_{gh}}\varphi_{gh} = 0;$$

(2) there exists in the m-space a vector ψ_{gh} such that

$$(T - w_gI)^{n_{gh}-\overline{n}_{gh}}\psi_{gh} = \varphi_{gh}.$$

33. If a function $g(x, y, z)$ can be put into the form of a determinant of third order the elements of each of whose rows depend on one variable only, the values of the variables which satisfy the equation $g(x, y, z) = 0$ may be read off from a figure by drawing a straight line across three curves (d'Ocagne, Traité de Nomographie, page 132). As it is not always easy to decide whether $g(x, y, z)$ can be given the determinant form, it is desirable to have conditions which permit us to answer this question. Dr. Gronwall in his valuable paper in the *Liouville's Journal*, series 6, volume 8, page 59, has given such conditions in the form of the existence of a common solution of two partial differential equations of second order. The criteria obtained in Professor Kellogg's paper demand for their application only the operations of differentiation and of the determination of the ranks of matrices. Incidentally differential conditions for the linear dependence of a set of functions of several variables are given. The paper will be offered to the *Zeitschrift für Mathematik und Physik.*

34. At the beginning of Professor Glenn's paper he shows that, if a function f which satisfies certain postulates can be expressed as a finite series in one argument A, then there always exists a function Φ and an operator Δ capable of being represented symbolically in the form $\Xi \cdot \partial/\partial/H$ such that

$$f = \Phi + \Delta\Phi A + \frac{\Delta^2}{\underline{|2}}\Phi A^2 + \cdots + \frac{\Delta^m}{\underline{|m}}\Phi A^m.$$

This is also stated in homogeneous form, and generalized for p arguments. In the second section is treated the problem of expressing a form a_x^{mn} in p variables, as a finite expansion of order m in p arguments A_i $(i = 1, 2, \cdots, p)$, through the intermediary of the relations $A_i = \alpha_{ix}^n$, $(i = 1, 2, \cdots, p)$. The coefficients in this expansion are irrational fractional expressions in the resultants of the various forms involved, and the operators Δ of the general theory are the Aronhold operators obtained from α_{ix}^n. In the third section a polynomial $\alpha_x^m (m = \mu\nu)$ is expanded in a series of powers of a form ϕ_x^ν. Here Φ and also Δ are determined in general. The last section contains a new derivation of the Clebsch-Gordan expansion.

35. Struve's investigations have revealed two cases of

libration, one of them of unusual magnitude, among the inner satellites of Saturn; so far only the most important librational terms in the longitudes of these satellites have been investigated. It is the aim of the present paper by Professor Laves to make a consistent study of the motions of all the four satellites as a whole and to determine the secular and librational inequalities in the eccentricities, perisaturnia, inclinations, and nodes, and to trace their influence on the longitudes of the satellites.

36. In a paper just published in *Crelle's Journal* (volume 142, page 165) Fejér has given two different limiting processes by each of which the magnitude of a finite jump of a function may be obtained from its Fourier development. There is a third method of doing the same thing which is such an immediate and obvious consequence of the generalized statement of Gibbs's phenomenon given by Professor Bôcher in the *Annals of Mathematics*, volume 7 (1906), page 131, that it may be said to be substantially contained in that statement. If $f(x)$ has at x_0 a finite jump of magnitude D, one obtains in this way the formula

$$D = G \lim_{n=\infty} \left[S_n \left(x_0 + \frac{2\pi}{2n+1} \right) - S_n \left(x_0 - \frac{2\pi}{2n+1} \right) \right],$$

where S_n denotes the sum of the first $n+1$ terms of the Fourier development of $f(x)$, and where

$$\frac{1}{G} = 1 - \frac{2}{\pi} \int_{\pi}^{\infty} \frac{\sin x}{x} dx.$$

This formula is similar to Fejér's formula (21), and perhaps even simpler than that formula. From it an analogue to his last formula on page 183 may at once be deduced. Very broad sufficient conditions on $f(x)$ for the correctness of these formulas are readily obtained, and generalizations are easily made.

37. In Science et Méthode, pages 158–160, Poincaré, in effect, constructs a general category of all inductive principles on the basis of resemblance. In this category occur the principle of "complete induction" and certain analogous

principles from which the former differs only by its certainty. The construction of the preceding general category depends itself upon the application of an inductive principle which seems not present in the category, The contradiction here presented by Mr. Schweitzer resembles the well-known contradiction of Richard.

38. The aims of Mr. Schweitzer's second paper are largely methodological: an attempt is made to describe the general logical position of mathematics as a heuristic science, to draw parallels between certain mathematical and philosophical authors, and to examine instances of the mathematical act. A definition of mathematics given by Peirce is adopted and is so interpreted as to recognize conflict as a part of mathematics. As leading examples of conceptual working hypotheses are discussed: (1) the principle of comparison, of which particular instances are the principles of Moore and Meinong; (2) the principle of continuation, of which a particular instance is the principle of permanence; (3) the principle of special situation; (4) the principle of the economy of thought and the working concepts beauty, elegance, simplicity, convenience, and naturalness. Leading examples of solutions of problems presented by conflicting terms in mathematics are mentioned, such as the "general analysis" of Moore, the "laws of thought" of Boole, the "extensive algebra" of Grassmann, etc. The main conclusions to which Mr. Schweitzer's investigation tends are stated thus: (1) The conceptual working hypotheses of mathematics are the same as those of non-mathematical disciplines; working hypotheses receive, however, a peculiarly mathematical character through discrimination exercised in their application. This discrimination is guided by perceptual working hypotheses: perceptual reconstructions of mathematical conceptions and primitive feelings or images. (2) There is essentially but one working hypothesis in mathematical procedure, viz., the principle of comparison.

39. If C_0 is the minimizing extremal for the calculus of variations problem

$$J = \int_{k_1}^{k_2} F(x, y, x', y')dt = \text{min.}, \quad K = \int_{k_1}^{k_2} G(x, y, x', y')dt = l$$

where k_1 and k_2 represent curves, and if C is any other curve connecting k_1 and k_2, satisfying the usual conditions, and for which $K = l$, the total variation is $J_{\bar{C}} - J_{\bar{C}_0}$. It is the object of Dr. Crathorne's paper to express this total variation in a form somewhat analogous to the Weierstrassian E-function representation for the simple calculus of variations problem.

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CONCERNING TWO RECENT THEOREMS ON IMPLICIT FUNCTIONS.

BY DR. LLOYD L. DINES.

(Read before the American Mathematical Society, October 26, 1912.)

THE theorems here considered are two recent generalizations of the Weierstrassian implicit function theorem,* by Professor G. A. Bliss† and Mr. G. R. Clements.‡ They will be referred to respectively as Theorem B, and Theorem C.

The two theorems are similar in that they both give information concerning the number and character of the solutions of a system of equations

$$(1) \qquad f_i(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0 \quad (i = 1, 2, \cdots, p)$$

in the neighborhood of a point at which the functional determinant vanishes. They are different in that the assumptions concerning the functions f_i are different. As is so often the case with similarly related theorems, the ranges of applicability overlap, but neither is wholly contained in the other.§ The purpose of this note is to characterize explicitly the four classes of cases: (I) in which neither theorem is applicable; (II) in which both theorems are applicable; (III) and (IV) in which one theorem is applicable while the other is not.

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* Weierstrass, Abhandlungen aus der Funktionenlehre, p. 107.

† Bliss, "A generalization of Weierstrass' preparation theorem for a power series in several variables," *Transactions*, vol. 13, pp. 133–45 (April, 1912).

‡ Clements, "Implicit functions defined by equations with vanishing Jacobian" (Theorem IV), BULLETIN, vol. 18, p. 453 (June, 1912).

§ In presenting this note to the Society, I made the statement that Mr. Clements's theorem was a corollary of Professor Bliss's. That this statement was incorrect was pointed out to me by Mr. Clements, who exhibited a numerical example in which the hypothesis of his theorem was satisfied while that of Professor Bliss was not. The example comes under Case IV as treated in this paper.