THE TWENTIETH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE annual meeting of the Society was this year held in New York City, on Tuesday and Wednesday, December 30–31, extending through the usual morning and afternoon sessions on each day. The attendance included the following eighty-one members:

Mr. E. S. Allen, Professor C. S. Atchison, Dr. Ida Barney, Mr. R. D. Beetle, Professor G. D. Birkhoff, Professor C. L. Bouton, Professor Pierre Boutroux, Professor Joseph Bowden, Dr. C. E. Brooks, Professor E. W. Brown, Mr. R. W. Burgess, Professor B. H. Camp, Dr. Emily Coddington, Dr. A. Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Dr. C. F. Craig, Dr. W. H. Cramblet, Professor G. H. Cresse, Professor L. P. Eisenhart, Professor G. C. Evans, Professor F. C. Ferry, Professor J. C. Fields, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor T. M. Focke, Dr. W. C. Graustein, Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. C. Grove, Professor J. G. Hardy, Professor H. E. Hawkes, Professor L. A. Howland, Professor L. S. Hulburt, Professor E. V. Huntington, Dr. W. A. Hurwitz, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. K. Lamond, Professor W. R. Longley, Professor C. R. MacInnes, Professor James Maclay, Professor H. W. March, Mr. B. E. Mitchell, Dr. H. H. Mitchell, Dr. F. M. Morgan, Professor Frank Morley, Professor Richard Morris, Mr. G. W. Mullins, Professor G. D. Olds, Professor W. F. Osgood, Dr. F. W. Owens, Dr. Anna Pell, Professor James Pierpont, Professor A. D. Pitcher, Professor Arthur Ranum, Dr. H. W. Reddick, Professor R. G. D. Richardson, Mr. L. B. Robinson, Mrs. E. D. Roe, Jr., Professor W. P. Russell, Professor L. P. Siceloff, Dr. Clara E. Smith, Mr. F. H. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. D. Thompson, Professor J. N. Van der Vries, Mr. H. S. Vandiver, Mr. J. N. Vedder, Mr. H. E. Webb, Professor H. S. White, Dr. E. E. Whitford, Professor E. B. Wilson, Professor W. A. Wilson, Professor Ruth G. Wood.

This meeting was especially marked as the occasion of the delivery by Professor H. B. Fine of his presidential address, on "An unpublished theorem of Kronecker respecting numerical equations."

Ex-President W. F. Osgood occupied the chair at the opening session, yielding it later to ex-President Fine. The Council announced the election of the following persons to membership in the Society: Professor Pierre Boutroux, Princeton University; Mr. E. H. Clarke, Purdue University; Dr. W. H. Cramblet, University of Rochester; Mr. H. J. Ettlinger, University of Texas; Professor W. S. Franklin, Lehigh University; Mr. Haig Galajikian, Princeton University; Professor W. W. Hart, University of Wisconsin; Mr. Barnem Libby, University of Michigan; Mr. G. W. Mullins, Columbia University; Mr. J. A. Northcott, Columbia University; Dr. Mildred L. Sanderson, University of Wisconsin; Mr. J. M. Stetson, Princeton University. Nine applications for membership in the Society were received.

The total membership of the Society is now 703, including 69 life members. The total attendance of members at all meetings of the past year was 418; the number of papers read was 240. The number of members attending at least one meeting during the year was 237. At the annual election 203 votes were cast. The Library now contains 4,902 volumes, excluding unbound dissertations. The Treasurer's report shows a balance of $9,153.58, including the life membership fund of $4,800.82. Sales of the Society's publications during the year amounted to $2,111.45.

A proposed amendment to the Constitution was submitted by the Council, making the secretary of the Chicago Section an ex-officio member of the Council. Final action on this amendment will be taken at the February meeting.

The annual dinner on Tuesday evening, attended by forty-seven members, was as always a most enjoyable occasion.

At the annual election, which closed on Wednesday morning, the following officers and other members of the council were chosen:

Vice-Presidents,	Professor L. P. Eisenhart,
	Professor E. J. Wilczynski.
Secretary,	Professor F. N. Cole.
Treasurer,	Professor J. H. Tanner.
Librarian,	Professor D. E. Smith.

Committee of Publication,
Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to serve until December, 1916,

Professor C. N. HASKINS, Professor E. V. HUNTINGTON,
Professor L. M. HOSKINS, Professor H. L. RIETZ.

The following papers were read at the annual meeting:

(1) Professor L. L. DINES: "Complete existential theory of Sheffer's postulates for Boolean algebras."

(2) Professor ARNOLD EMCH: "Two convergency proofs."

(3) Professor J. L. COOLIDGE: "Congruences and complexes of circles."

(4) Dr. DUNHAM JACKSON: "On the degree of convergence of Sturm-Liouville series."

(5) Professor VIRGIL SNYDER: "Birational transformations of the cubic variety in four-dimensional space."

(6) Miss A. H. TAPPAN: "Plane sextic curves invariant under a group of linear transformations" (preliminary communication).

(7) Professor C. L. BOUTON: "Explicit formulas for the inverse of an analytic transformation in n variables."

(8) Professor EDWARD KASNER: "The classification of conformal transformations."

(9) Mr. L. B. ROBINSON: "Questions of logic arising from the study of systems of partial differential equations" (preliminary report).

(10) Professor PIERRE BOUTROUX: "On a family of rational differential equations of the first order."

(11) Professor H. B. FINE, Presidential Address: "An unpublished theorem of Kronecker respecting numerical equations."

(12) Dr. W. A. HURWITZ: "Note on the Fredholm determinant."

(13) Professor G. D. BIRKHOFF: "The restricted problem of three bodies."

(14) Professor E. V. HUNTINGTON: "On the accuracy of the contracted form of Horner's method."

(15) Professor O. E. GLENN: "On an analogy between formal-modular invariants and the class of algebraical invariants called Booleans."

(16) Professor G. C. EVANS: "Green's functions for linear partial differential expressions of the second order, and Green's theorem."

(17) Professor W. R. LONGLEY: "An existence theorem for a certain differential equation of the nth order."

(18) Dr. W. C. GRAUSTEIN: "The real-congruence of complex points, planes, lines."

(19) Dr. H. W. REDDICK: "Conformal invariants of an orthogonal curve net" (preliminary communication).

(20) Professor W. F. OSGOOD: "Liouville's theorem concerning periodic functions of several variables."

(21) Mr. L. M. KELLS: "Complete characterization of dynamical trajectories in n-space."

(22) Dr. W. H. CRAMBLET: "A classification of discontinuous functions and some allied problems."

(23) Dr. J. K. LAMOND: "Note on the reduction of multiple L-integrals to iterated L-integrals."

(24) Professor L. A. HOWLAND: "Functions of n variables which are functions of r combinations of these variables."

(25) Dr. J. I. TRACEY: "Covariant curves of the plane rational quintic."

(26) Professor A. B. COBLE: "Restricted systems of equations."

(27) Professor L. P. EISENHART: "Transformations of surfaces of Voss."

(28) Mr. H. GALAJIKIAN: "A type of non-linear integral equation."

(29) Dr. T. H. GRONWALL: "On systems of linear total differential equations."

(30) Dr. T. H. GRONWALL: "Extension of Laurent's theorem to several variables."

(31) Dr. T. H. GRONWALL: "On approximation by trigonometric sums."

(32) Mr. R. D. BEETLE: "Cyclic systems of osculating circles of curves on a surface."

(33) Dr. G. M. GREEN: "Canonical systems in projective differential geometry, with special reference to the theory of curved surfaces."

(34) Professor J. H. M. WEDDERBURN: "A type of primitive algebra."

Miss Tappan's paper was communicated to the Society through Professor Snyder. Mr. Kells was introduced by

Professor Kasner. In the absence of the authors the papers of Professor Dines, Professor Emch, Miss Tappan, Professor Glenn, Professor Evans, Mr. Galajikian, and Professor Wedderburn were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In the October number of the *Transactions* Dr. Sheffer presented an elegant and concise set of five postulates for Boolean algebras, and proved them mutually consistent and independent. In the present paper, Professor Dines shows that these postulates, though independent in the ordinary sense that no four of them imply a fifth, are not *completely independent* in the sense in which that term has been used by Professor E. H. Moore.* A "complete existential theory" is constructed for the five postulates, and it is found that there exists among the postulates and their negatives one and only one general implicational relation, namely that the negative of the first implies the third, fourth, and fifth.

2. In the study of automorphic functions defined within a fundamental domain formed by two non-intersecting circles in the elliptic case and by two tangent circles in the trigonometric case, it is necessary to prove the convergency of certain fundamental series, as has been done by Schottky recently.† The first is the series Σr_λ of the radii of all circles forming the boundaries of the domains belonging to the cyclic loxodromic group. The second is the series

$$\Sigma(x_\lambda - x_\lambda')$$

for the cyclic parabolic group, in which x_λ and x_λ' result from x and x' by applying to them λ times in succession the original substitution. Professor Emch gives direct proofs for the convergence of both series.

3. A circle in space may be expressed parametrically by equating the pentaspherical coordinates of a point to five

* E. H. Moore: "An introduction to a form of general analysis," New Haven Mathematical Colloquium, Yale University Press, p. 82.

† "Ueber eine Funktionenklasse, die der Gleichung $F\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) = F(x)$ genügt," *Journal für reine und angewandte Mathematik*, vol. 143, pp. 1–24 (May, 1913). See also the same *Journal*, vol. 101, pp. 231–236 (1887).

quadratic functions of an auxiliary variable. In the first part of Professor Coolidge's paper it is shown how this leads to easy and uniform proofs of all those classic theorems about circles which are invariant for inversion. In the second part of the paper the circle is treated as the envelope of the spheres through it, and by an adaptation of the Kummer methods for line geometry several new types of circle congruence are brought to light and discussed. In the third part the same methods are extended to the study of complexes of circles.

4. Dr. Jackson's paper gives the extension of theorems on the degree of convergence of Fourier's series to more general series arising from a linear homogeneous differential equation of the second order, the differential equation and the boundary conditions being those discussed by Kneser in volume 58 of the *Mathematische Annalen.* Kneser, following Liouville, makes use of relations which, when applied to the present problem, render the extension immediate in the simplest cases, and suggest the method of treatment in less simple ones. If the function to be developed has a continuous kth derivative of limited variation, or a $(k - 1)$th derivative satisfying a Lipschitz condition, certain other restrictions being imposed in each case, the remainder after n terms of the series does not exceed a constant multiple of $1/n^k$ or of $(\log n)/n^k$ respectively. It is shown also that a method of summation previously applied to render the Fourier series more rapidly convergent in certain cases is equally applicable to the Sturm-Liouville series. For example, if the function developed satisfies a Lipschitz condition, it is possible to represent it approximately by a linear combination of the first n characteristic functions with a maximum error not exceeding a constant multiple of $1/n$.

5. Various methods have been employed to determine whether the three-dimensional cubic variety V in linear space of four dimensions can be birationally mapped on ordinary space S_3. Professor Snyder here attempts to determine whether continuous groups of birational transformations exist which leave V invariant. All such transformations that are known to exist, while forming a continuous series, do not form a continuous group.

The operations considered are central projections of V into itself from points on it, and the polar transformation, defined as follows: On V lies a two-dimensional series of straight lines; through each point pass six. Associated with each line is a two-dimensional surface, locus of the pole of the line as to the residual conic in each plane passing through it. To find the image of any point P, pass a plane through P and the line, and join P to the pole of the line. The harmonic conjugate of P as to the conjugate points as to the conic is the image. Both transformations are Cremonian and involutorial, but the product of any two or more is not periodic.

6. In Miss Tappan's paper three types of linear transformations are considered: (a) Those having the vertices of a triangle for invariant points; (b) the permutation group of the three homogeneous point coordinates; (c) those not of types (a) and (b), including the G_{168} and G_{360}. Only one sextic can have a continuous group of linear transformations. The highest cycle in (a) is of order 30; the curve belonging to it is of genus 10. Sixty-three distinct types of curves with groups of form (a) exist. For a cycle of order 21 and one of order 12, the corresponding curve must also have transformations of type (b). The Cremona and Riemann non-linear birational transformations of sextic curves are being studied, but the results are not yet completed.

7. Consider the special transformation

$$(1) \qquad x_i' = \phi_i(x_1, \cdots, x_n) \equiv x_i + \sum_{k=2}^{\infty} \phi_{ik}(x_1 \cdots, x_n)$$

$$(i = 1, 2, \cdots, n),$$

where the given functions $\phi_i(x)$ are analytic at the origin, so that the series $\Sigma\phi_{ik}$ is a convergent power series, and ϕ_{ik} is a homogeneous polynomial of degree k. For the neighborhood of the origin the inverse of (1) will then have the same form, viz.,

$$(2) \qquad x_i = \phi_i'(x_1', \ldots, x_n') \equiv x_i' + \sum_{k=2}^{\infty} \phi_{ik}'(x_1', \cdots, x_n')$$

$$(i = 1, 2, \cdots, n),$$

where the power series $\Sigma\phi_{ik}'$ are convergent. Professor Bouton's paper gives the explicit formulas for the computation of the ϕ_{ik}' in terms of the ϕ_{ik}. We have

$$(3)\quad \phi_{i2}'(x) \equiv -\phi_{i2}(x), \quad \phi_{ik}'(x) \equiv -\phi_{ik}(x) + \sum_{j=1}^{k-2} Q_j \phi_{i,k-j}$$

$$(k = 3, 4, \cdots),$$

where Q_j is a homogeneous linear differentiating operator which, applied to a homogeneous polynomial, raises its degree j units. These operators are found successively from the symbolic identity

$$(4)\quad \begin{aligned} \sum_1^\infty Q_j \equiv &-\sum_{t=1}^n \psi_t' \frac{\partial}{\partial x_t} - \frac{1}{2!}\sum_{s=1}^n\sum_{t=1}^n \psi_s'\psi_t' \frac{\partial^2}{\partial x_s \partial x_t} \\ &- \frac{1}{3!}\sum_r\sum_s\sum_t \psi_r'\psi_s'\psi_t' \frac{\partial^3}{\partial x_r\partial x_s\partial x_t} - \cdots \end{aligned}$$

by equating operators of like degree, where $\psi_t' \equiv \sum_{k=2}^\infty \phi_{tk}'$. For example,

$$Q_1 = \sum_{t=1}^n \phi_{t2}\frac{\partial}{\partial x_t}, \quad Q_2 = \sum_{t=1}^n (\phi_{t3} - Q_1\phi_{t2})\frac{\partial}{\partial x_t} - \frac{1}{2!}\sum_s\sum_t \phi_{s2}\phi_{t2}\frac{\partial^2}{\partial x_s\partial x_t}.$$

The formula for Q_j is readily written down, and thus (3) gives the inverse of (1).

Finally, it is pointed out that any analytic transformation in the neighborhood of a regular point is projectively equivalent to the form (1), so that we have formulas for the inverse of any such transformation.

An application is given to the case of Cremona transformations.

8. In Professor Kasner's classification two conformal transformations (assumed regular at a given point) are regarded as equivalent when the one can be transformed into the other by a conformal transformation. In previous papers the author has discussed (conformal) invariants of analytic curves and of curvilinear angles. The present problem is to find invariants of a conformal transformation $w = f(z)$. The results are closely related to the previous invariants, though the geometric interpretations are quite distinct. The only obvious invariant is w', the value of the first derivative of $f(z)$ at the given point. The simplest example of a higher invariant arises for the class of transformations characterized by $w' = 1$, $w'' \neq 0$, the result being w'''/w''^2, of the third

order. If the transformation has a singularity at the given point, that is, if $w' = 0$, then it can be reduced to the normal form $w = z^k$ $(k > 1)$, so that no differential invariants exist.

9. In one of his papers Riquier has stated that a reduction of a system of partial differential equations to the first order would be advantageous. Mr. Robinson discusses what these advantages might be, making use of the principle of "correlation multiplicatoire." The reduction of a system to the orthonomic passive form, which has been left in an incomplete state by Riquier, is taken up in some detail.

10. The object of Professor Boutroux's paper is to distinguish a special family of rational differential equations of the first order and the first degree, and to show how the integrals of these equations can be investigated in respect to their general behavior (croissance), the number and the situation of their critical points, etc., and how the different branches of these integrals can be calculated by a process of successive approximations and represented by a single convergent series in the whole plane of the variable.

The simplest example of a differential equation of the family in question is provided by the equation

$$\frac{dy}{dx} = A_0 + A_1 y + A_2 y^2 + A_3 y^3,$$

which may be written

$$z\frac{dz}{dx} = A_3 + A_2 z + A_1 z^2 + A_0 z^3, \tag{1}$$

and which Professor Boutroux has already considered in several papers. Equation (1) belongs to our family when A_1 and A_0 are both equal to zero identically and the degrees m_3 and m_2, in x, of A_3 and A_2 satisfy the condition $m_3 > 2m_2 + 1$.

If we now consider the general rational equation of the first order and degree, it may be written

$$\frac{dz}{dx} = \frac{A_0 + A_1 z + \cdots + A_p z^p}{B_0 + B_1 z + \cdots + B_q z^q}, \tag{2}$$

where the A's and B's are polynomials in x, the respective degrees of which we shall call $m_0, \cdots, m_p$ and $n_0, \cdots, n_q$.

Equation (2) will belong to our family if the following conditions are satisfied:

$$p < q + 1; \quad m_0 - n_{q+1} > 0;$$

and

$$m_1 + \sigma < m_0, \cdots, m_p + p\sigma < m_0;$$

$$n_0 + \sigma - 1 < m_0, \cdots, n_{q-1} + q\sigma - 1 < m_0;$$

moreover, we suppose that there is no common root of the q polynomials $B_0, \cdots, B_q$.

11. Professor Fine's presidential address will appear in full in the April BULLETIN.

12. Dr. Hurwitz gives another proof of a theorem recently announced by Platrier (Paris thesis), which expresses minors of any order of the Fredholm determinant for the kernel of an integral equation in terms of the first minor and the determinant itself.

13. By means of a simple explicit reduction of the restricted problem of three bodies to the consideration of the transformation of a ring into itself, Professor Birkhoff establishes the existence of infinitely many symmetric orbits for certain ranges of values of the Jacobian constant, and orders these in four categories, each member of any category depending on two characteristic integers. A number of other results are also obtained.

A ring representation, but in Keplerian variables, was one of the great steps in advance accomplished by Poincaré (Les Méthodes nouvelles de la Mécanique céleste, volume 3, chapter 33). In a recent paper (*Rendiconti del Circolo Matematico di Palermo,* volume 33 (1912), pages 375–407) he indicated, on the basis of this representation, that if a certain geometric theorem (established by Professor Birkhoff, *Transactions,* volume 14 (1913), pages 14–22) were true, the existence of an infinite number of periodic solutions would follow for the above and similar problems; this fact had not been established by Poincaré in his earlier work, although it had been proved by him that for small enough values of one of the masses the number of periodic solutions was arbitrarily large.

Professor Birkhoff's present paper does not employ the geometric theorem of Poincaré but employs instead a certain symmetry, in consequence of which the transformation of the ring is seen to be a product of two involutoric transformations.

14. In the contracted form of Horner's method, as described, for example, in Burnside and Panton's Theory of Equations, the contraction may be commenced at any time after the decimal portion of the root begins to appear, and the number of additional figures obtained by the contracted process will be one less than the number of figures in the trial divisor at the time the contraction commences. The question as to how many of the additional figures thus obtained will be reliable seems not to be discussed in any of the current text-books. Professor Huntington shows that the last n figures of the root as thus obtained may be erroneous, where n is the degree of the equation. As an example, he constructs an equation of the sixth degree, having the following coefficients: $a_0 = 500{,}000$, $a_1 = -3{,}295{,}000$, $a_2 = 9{,}047{,}550$, $a_3 = -13{,}249{,}720$, $a_4 = 10{,}914{,}563$, $a_5 = -4{,}794{,}993$, $a_6 = 877{,}562$. This equation has a root between 1 and 2. If we obtain the first two figures by the unabbreviated process and then as many more as possible by the contracted process, we shall find for the root the value 1.19 160 438. The last six figures in this result are wrong. The paper will be offered to the *American Mathematical Monthly*.

15. Professor Glenn considers invariants and covariants of binary forms whose coefficients are arbitrary variables, subject to transformations which are of the general binary linear type but whose coefficients are parameters representing integers reduced modulo p. These invariants were first defined by Hurwitz. The present paper deals with a method of finding systems of concomitants of this type. This is done by combining methods of Boole for systems now called Boolean, with results by Dickson in his memoir, "A fundamental system of invariants, etc.," *Transactions*, volume 12.

The paper contains tables of irreducible concomitants of the binary quadratic modulo 3, and of the binary cubic modulo 2. These concomitants are exhibited as transvectants taken with respect to the modulus. A section on general properties and the annihilators of this type of invariant is also included in the paper.

It is not yet known whether the formal modular concomitants of a form, as described above, constitute a closed system.

16. It is Professor Evans's object to express Green's function for the general linear partial differential expressions of the second order, in terms of Green's functions for the differential expressions consisting of the terms of the second order alone. These functions may be obtained by means of a system of linear integral equations, whose solutions may be written in closed form, according to a method first used by Hilbert. To carry out the analytical work effectively, however, it is desirable to consider these terms of the second order as a single differential operator, rather than as the sum of derivatives. This is, in fact, a point of view suitable to the applications in physics.

Such an interpretation is given by Professor Bôcher for the special case of Laplace's equation ("Harmonic functions in two dimensions," *Proceedings of the American Academy of Arts and Sciences*, volume 41). By means of this convention and a slight extension of the usual form of Green's theorem, the boundary value problems for the *non-homogeneous* equations may be solved without demanding so much as the finiteness and integrability of the non-homogeneous terms. The results for the non-homogeneous equations may then be applied to the solution of integro-differential equations of various types.

17. The differential equation considered by Professor Longley is $d^ny/dx^n = N/D$, where N and D are convergent integral power series in x and y, vanishing when $x = y = 0$. Similar singularities for a system of n equations of the first order have been treated by several writers* and from this general theory some results may be deduced concerning the equation above. In the present paper these results are proved directly and extension is made to some cases for which the general theory gives no information.

18. Dr. Graustein discusses the conditions under which two complex elements (points, planes, lines) of three-dimensional complex space may be real-congruent, i. e., equivalent under the group of real motions. The case of complex points

* An exposition of the results obtained, with some simplifications in method, is given by Dulac, *Bulletin de la Société mathématique de France*, vol. 40 (1912), pp. 324–383. The article contains references to several earlier papers.

is simple. Necessary though not in every case sufficient conditions for the real-congruence of two complex planes (or lines) result from the classification of complex planes (or lines) according to the number and situation of their real points and also according to whether or not they are minimal. To obtain necessary and sufficient conditions a complete system of absolute rational invariants of the complex plane (or line) in respect to the group of real motions is developed; the desired conditions may be stated in terms of these invariants. At the end of the paper the invariants of each complex element in respect to other real groups are discussed and from this the equivalence of two like named complex elements under these groups.

19. Dr. Reddick uses the power series method employed by Professor Kasner in dealing with conformal invariants of curvilinear angles (Proceedings of the Fifth International Congress of Mathematicians, volume 2, page 81). The transformations $Z = a_1z + a_2z^2 + \cdots$, where the coefficients are real, leave the X-axis invariant and form a subgroup of the entire conformal group. A classification of the invariants of an orthogonal net under this subgroup is given. Certain combinations of these invariants with those under the subgroup leaving the Y-axis invariant give invariants of the net under the entire conformal group.

20. Professor Osgood shows that Liouville's theorem, that a doubly periodic function of a single variable which is analytic for all finite values of the arguments is a constant, admits the following generalization:

Let $F(z_1 \cdots, z_n)$ be an integral function having in $(P_1^{(k)}, \cdots, P_n^{(k)})$, $k = 1, \cdots, r$, a primitive system of periods. Let the rank of the matrix

$$\begin{Vmatrix} P_1' & \cdots & P_1^{(r)} \\ \cdot & \cdot & \cdot \\ P_n' & \cdots & P_n^{(r)} \end{Vmatrix}$$

be denoted by ρ. Then $r \leqq \rho$.

A second theorem intimately related to the foregoing is as follows:

Let $(P_1^{(k)}, \cdots, P_n^{(k)})$, $k = 1, \cdots, r$, be r systems of complex quantities not satisfying a linear relation with real coefficients

$$c'P_l' + c''P_l'' + \cdots + c^{(r)}P_l^{(r)} = 0,$$

$$l = 1, \cdots, n, \text{ and } |c'| + \cdots + |c^{(r)}| > 0.$$

Let the rank of the matrix

$$\begin{Vmatrix} P_1' & \cdots & P_1^{(r)} \\ \cdot & \cdot & \cdot \\ P_n' & \cdots & P_n^{(r)} \end{Vmatrix}$$

be denoted by ρ. Let $F(z_1, \cdots, z_n)$ be an integral function admitting the r periods $(P_1^{(k)}, \cdots, P_n^{(k)})$. If $r > \rho$, then F is a constant.

21. There is given in n-space an arbitrary field of force in which the motion of a freely moving particle is defined by the equations

$$\frac{d^2x_i}{dt^2} = \psi_i(x_1, x_2, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

where the ψ_i are arbitrary functions of the n variables $x_1, \cdots, x_n$, possessing derivatives of the first and second orders in the region of space considered.

The object of Mr. Kells' paper is to find a set of purely geometric properties which completely characterize the system of curves belonging to the above defined field of force. The following four properties are found to be sufficient:

I. The osculating planes at a given point of all curves of the system through that point form a hyperpencil of planes in n-space, that is, all osculating planes at a point pass through a fixed line.

II. The centers of the third order osculating hyperspheres at a point of the ∞^1 curves passing through that point in a given direction lie in an $(n - 2)$-flat.

III. The ∞^{n-1} $(n - 2)$-flats which correspond according to property II to the ∞^{n-1} lineal elements at a point make up the secant $(n - 2)$-flats of an nth order n-space curve which passes through the point in the direction of the axis of the hyperpencil of osculating planes at the point.

IV. Certain plane systems of curves associated with the $(2n - 1)$-fold infinite system of curves in n-space are of the two-dimensional dynamical type.

(Professor Kasner has given the discussion for the cases $n = 2$ and $n = 3$. See *Transactions* for 1906 and for 1907.)

22. If one replace the E sets of the function $f(x)$, as defined by M. Lebesgue, by integrand sets similar to those used by W. A. Wilson, it is shown that these upper and lower functions must exist between every Baire class. In fact, their existence between every Baire class is a necessary and sufficient condition for the existence of functions of every Baire class. By the use of these intermediate functions it is possible to extend Baire's theorem on pointwise discontinuous functions to functions of every Baire class. Dr. Cramblet also proved that a function of class n defined over any set $\mathfrak{A}$ may be extended over a set $\mathfrak{B} > \mathfrak{A}$ without altering the class of the function. Finally, necessary and sufficient conditions that the limit of a sequence of functions of any class be a function of that class are proved, and are found to include Arzelà's convergence in segments as a special case.

23. In this note Dr. Lamond gives sufficient conditions for the existence and equality of the multiple and the iterated L-integrals of a function, over a field which may not be measurable.

24. The existence of certain relations between the first partial derivatives of a function of n independent variables is shown by Professor Howland to be a necessary and sufficient condition that it be a function of r combinations of these variables ($r < n$). The case where these combinations of the variables are linear is discussed at greater length. Two conditions for this case are derived, one involving the rank of a determinant, the other the rank of a matrix, and relations are shown to exist between the determinant and certain determinants of the matrix.

25. In Dr. Tracey's paper the plane rational curve of order n is given by the equation $(a\xi)(\alpha t)^n = 0$ and this is treated as a binary n-ic in the parameter t of the curve. Any covariant of this form involves the coordinates of the line ξ, and for a varying ξ becomes a class curve. If a covariant vanishes identically, a system of class curves is obtained all of which contain special line sections of the base curve, and in some cases the complete system of curves of a given order on these lines is at once determined.

A number of these covariant curves are discussed for the rational quintic. The paper will be published in the *American Journal of Mathematics.*

26. The theory of restricted systems of equations has not been developed thus far in such a way as to permit of its application to many of the relatively simple problems of enumeration in geometry and algebra. The object of Professor Coble's paper is to remodel this theory and to increase thereby its utility.

27. A surface of Voss is characterized by the property that there exists upon it a one-parameter family of geodesics whose conjugate curves are geodesics also. This conjugate system has the same Gaussian representation as the asymptotic lines of a pseudospherical surface. Professor Eisenhart has established a transformation of surfaces of Voss which has the following geometrical properties: If S_1 is a surface resulting from S by such a transformation, the lines joining corresponding points of these surfaces form a congruence whose developables cut S and S_1 in the geodesic conjugate system. Moreover, the line of intersection of the tangent planes to S and S_1 at corresponding points generates a normal congruence, whose focal planes bisect the angles between these tangent planes; and the pseudospherical surfaces with the same spherical representation of their asymptotic lines as the conjugate system on S and S_1 can be brought into the relation of the transformation of Bäcklund. The analytical determination of the transformations requires the solution of a Riccati equation and quadratures. With each pseudospherical surface there are associated in the manner mentioned above a family of *special* surfaces of Voss, each of which admits a transform such that the lines joining corresponding points on the two surfaces are concurrent. These special surfaces play an important role in the general transformations of surfaces of Voss. It may be shown also that if S_1 and S_2 are two transforms of a surface of Voss S there exists a fourth surface S' which is a transform of both S_1 and S_2; moreover, S' can be found without quadratures.

28. In showing that the first derivatives of a certain nonlinear integral equation exist, Mr. Galajikian was led to the consideration of an equation of the type

$$u(x, x_0) = g\left\{x, \int_{x_0}^{x} f_1\left(x, t, \int_{x_0}^{t} u(t, \xi)d\xi\right)u(t, x_0)dt,\right.$$
$$\left.\int_{x_0}^{x} f_2\left(x, t, \int_{x_0}^{t} u(t, \xi)d\xi\right)u(t, x_0)dt\right\}.$$

In the present paper he proves, by a modification of Picard's approximation method, that a unique solution of this new equation exists in a sufficiently small interval.

29. In this paper, Dr. Gronwall considers systems of linear total differential equations in n independent variables such that their general solution depends linearly on a finite number of particular solutions. The coefficients of the system are assumed to be meromorphic. After a discussion of the decomposition of the singularities of the coefficients into irreducible manifolds, the main results of Fuchs's theory of linear differential equations are extended to the present case. Letting $\varphi(x_1, \cdots, x_n) = 0$, where φ is an entire function, be one of the irreducible singular manifolds, it is shown that each of a fundamental system of solutions may be expressed in the form: φ^r times a polynomial in $\log \varphi$, the coefficients of which are Laurent series of the kind considered in the following paper. The reduction of the system of equations to a canonical form is then taken up and applied to finding the necessary and sufficient conditions that the Laurent series mentioned shall contain only a finite number of negative powers of φ.

30. Dr. Gronwall obtains an extension of Laurent's theorem, which may be stated as follows, in the simplest case: Let $f(x_1, \cdots, x_n)$ be a uniform analytic function, holomorphic at all finite points except those satisfying the equation $\varphi(x_1, \cdots, x_n) = 0$, where φ is an entire function. Then

$$f(x_1, \cdots, x_n) = \sum_{\lambda=-\infty}^{\lambda=\infty} \frac{\psi_\lambda(x_1, \cdots, x_n)}{[\varphi(x_1, \cdots, x_n)]^n},$$

the ψ's being entire functions, and this series is uniformly convergent in any finite region, the points of which satisfy the inequality $|\varphi(x_1, \cdots, x_n)| > \epsilon > 0$. It is, however, essential to note that the above development is not unique, as in the case of one variable.

31. Dr. Jackson has shown (*Transactions*, October, 1912)

that, $f(x)$ being a function of period 2π and satisfying the Lipschitz condition

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|$$

for all values of x_1 and x_2, then there exists, for every integer n, a trigonometric sum of order m $(2m - 2 \leqq n \leqq 2m)$,

$$\begin{aligned} T_n(x) = a_0 &+ a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx \\ &+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx \end{aligned}$$

such that for all values of n

$$|f(x) - T_n(x)| \leqq \frac{J_m'}{J_m} \cdot \frac{\lambda}{n}.$$

Here J_m' and J_m are certain definite integrals, the quotient of which is bounded for all values of m, and by asymptotic considerations, Dr. Jackson finds an upper boundary for this quotient. In the present paper, Dr. Gronwall gives a direct proof that the quotient in question decreases as the integer m increases, and thus obtains a closer upper boundary.

32. The paper by Mr. Beetle establishes the following theorem:

In order that the osculating circles of a one-parameter family of curves on a surface form a cyclic system, it is necessary and sufficient that the curves be lines of curvature of constant geodesic curvature. If the geodesic curvature is zero, the curves are the plane lines of curvature of a surface of Monge. If the geodesic curvature is different from zero, the curves are the spherical lines of curvature on one of the surfaces S obtained from the surfaces of Monge by inversion, or on one of the surfaces obtained by subjecting the surfaces S to Bianchi's Combescure transformation* of surfaces with a system of spherical lines of curvature.

33. In the study of certain configurations by means of Professor Wilczynski's general method, the discussion is much simplified by throwing the general system of differential equations S into a canonical system S'. The reduction frequently requires the integration of partial differential equations, and so is of course (practically) impossible. Dr. Green

* Bianchi, Lezioni di Geometria differenziale, vol. 2, p. 306.

illustrates, by setting up a complete system of invariants and covariants for the general system of differential equations S of the theory of curved surfaces, a fact of which he has made use in previous papers (see abstracts in the January BULLETIN, pages 171, 172). The invariants and covariants calculated for the system S' may be expressed explicitly in terms of the coefficients of S, without actually requiring the integration necessary to throw S into the form S'. Not only is this procedure preferable on account of the simplifications which result from the use of the canonical form S', but it also gives an insight into the real nature of the invariants and covariants.

34. In a recent paper* Professor L. E. Dickson has discussed the algebra defined by the relations

$$xy = y\theta(x), \quad y^n = g,$$

where $\theta(x)$ is a rational polynomial in x in a given field F, and y^n is the first power of y which is commutative with x. In this note Professor Wedderburn gives a simple proof that, if g is properly chosen in F, this algebra is primitive, i. e., division by any number of the algebra except zero is always possible.

F. N. COLE,
Secretary.

WINTER MEETING OF THE SOCIETY AT CHICAGO.

THE thirty-second regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, December 26–27, 1913. In accordance with a recent action of the Council the meetings of the Chicago Section for the presentation of scientific papers are hereafter to be designated as meetings of the Society at Chicago, and this was the first occasion to be observed under the new arrangement. Seventy-three persons were in attendance, including the following fifty-six members of the Society:

Professor W. H. Bates, Professor G. A. Bliss, Dr. Henry Blumberg, Professor Daniel Buchanan, Professor W. H. Butts, Professor H. E. Cobb, Professor D. R. Curtiss, Professor S. C. Davisson, Professor L. E. Dickson, Professor Arnold Dresden,

* *Transactions*, Jan., 1914.