A NON-ENUMERABLE WELL-ORDERED SET.

BY PROFESSOR A. B. FRIZELL.

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THE article " Intuitionism and formalism " in the November BULLETIN (pages 81–96) raises the question whether the field of denumerably infinite processes is a closed domain. The present note, based on a paper which the writer presented to the Cambridge Congress under the title "Axioms of ordinal magnitudes,"* undertakes to show that this question must be answered in the negative.

It is admitted by both formalist and intuitionist that the natural numbers rest on valid assumptions. In other words, it is legitimate to postulate two rules of combination—let us call them " lower " and " higher "—whereof the higher is to be distributive over the lower, a modulus for the higher rule, and the totality of symbols obtainable therefrom by the " axioms of numbering "† in the paper just mentioned.

Let us drop the postulate of the modulus, leaving λ with no properties except that it is to precede every other element of the set. The higher rule now generates automatically the ω-series

$$\lambda_2' = \lambda \square \lambda, \quad \lambda_2'' = \lambda \square \lambda_2', \quad \cdots,$$

while the lower rule yields not only a similar set

$$\lambda_1' = \lambda \bigcirc \lambda, \quad \lambda_1'' = \lambda \bigcirc \lambda_1', \quad \cdots,$$

but also other such sets built up on other members of the first

$$\lambda_{(2,1}^{(n,2} = \lambda_2^{(n} \bigcirc \lambda_2^{(n}, \quad \lambda_{(2,1}^{(n,3} = \lambda_2^{(n} \bigcirc \lambda_2^{(n,2}, \quad \cdots \quad (n = 1, 2, \cdots)$$

and sets of polynomial elements.

This process makes no use of the distributive law; the same result would be reached by postulating two rules totally unconnected except by the relation of order. And if it is per-

* Proceedings of the Fifth International Congress of Mathematicians, pp. 327–336.

† Loc. cit., p. 328. A numbered set is an ω-series of symbols generated by the law that if κ belongs to the set so does $\lambda \bigcirc \kappa$ (resp. $\kappa \bigcirc \lambda$) also, where λ is an "ausgezeichnetes Element." The set is ordered so that $\kappa < \lambda \bigcirc \kappa$, resp. $\kappa < \kappa \bigcirc \lambda$.

missible to postulate two rules entirely independent of one another, then we are also entitled to an ω-series of unrelated rules together with all symbols that can be generated from λ by applying these rules in accordance with the axioms of numbering. Such a set is postulated in the "transfinite axioms"* of the paper already quoted.

It is proposed to prove that this set of symbols cannot be put into one-to-one correspondence with the natural numbers. It is sufficient to consider polynomials like $\omega_{(10}^{(19} + \omega_{(9}^{(250} + w_{(8}^{(101} + \cdots + \omega_{(2}^{(83}$, i. e., those formed by the lowest rule and having every term a primitive element. Suppose this set denumerable so that we may write them: $P_1, P_2, \cdots, P_n, \cdots$. Denote by p_1 the lowest term in P_1, by p_2 the next to lowest term in $P_2, \cdots$, by p_n the nth lowest in P_n. Let the corresponding upper indices be $i_1, i_2, \cdots, i_n$ so that $p_n = \omega_{(\nu}^{(i_n}$ and write $i_n = 0$ if P_n has less than n terms. Form a new polynomial by changing these indices to $\kappa_1 \dotplus i_1, \kappa_2 \dotplus i_2, \cdots, \kappa_n \dotplus i_n$. Either it is possible in this way to get a polynomial not in the ω-series $P_1, P_2, \cdots$ or it is not. The first alternative contradicts the assumption of denumerability. The second implies polynomials with an actually infinite number of terms, admitting each an ω-series of upper indices. Such a "Belegung," however, is a continuum† and hence not countable. Thus in either case the supposition in false, and the theorem is proved.

Thus by proceeding strictly in accord with the demands of neo-intuitionism, using only the "bare two-oneness" to which this philosophy stands committed, we are inevitably led to a concept which it refuses to recognize. This, of course, need not disturb an intuitionist who cuts loose from the principium contradictionis, but it is matter for regret to find formalism attacked by aid of the principium on the grounds of the Burali-Forti "paradox," which is nothing more than the logical freak of the illusory self-contradictory concept.‡

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* Loc. cit., p. 329. These symbols fall into three classes: (1) *primitive* elements, of the type $\omega_{(n}^{(i}$; (2) *monomial* elements, given by the formula $\omega_{(\nu_1, \nu_2, \ldots \nu_n)}^{(i_1, i_2, \ldots i_n)}$ $(\nu_1 > \nu_2 > \cdots)$; (3) *polynomials*, or combinations of monomials by lower rules than those used in any individual term.

† There really is a continuum contained in the set of symbols postulated by axioms T^{ω_1} of the paper previously cited (p. 333), since every element of this set involves an actual infinity of "Bestimmungstücke."

‡ For example, "the totality of self-contradictory concepts" is a self-contradictory concept, because it does not include the concept "a self-contradictory concept not contained in the totality of self-contradictory concepts"; but the latter is a self-contradictory concept.