of rigor, and its clever handling of a pedagogically difficult subject should recommend it to teachers of mathematics.

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Vorlesungen über Variationsrechnung. Von OSKAR BOLZA. Leipzig, B. G. Teubner, 1909. ix+795+10 pp.

THIS book is a revised and considerably enlarged German edition of the same author's "Lectures on the Calculus of Variations," Chicago, University Press, 1904.*

In Chapter I, entitled "The first variation in the simplest class of problems," the author, after some introductory remarks on the scope of the calculus of variations, starts by explaining his system of notations, which is exceedingly precise and consistent, although it would seem to the reviewer that a somewhat less elaborate system would have made the book easier to read without any sacrifice of rigor. The classical results in the theory of the first variation of the integral $\int_{x_1}^{x_2} f(x, y, y')dx$ with fixed and variable end points are set forth, including Euler's differential equation and Du Bois-Reymond's lemma. The proof for the latter given on page 28 is due to Hilbert; it would perhaps have been more appropriate to give the proof of Zermelo (*Mathematische Annalen*, volume 58 (1904), page 558), which is unsurpassed in simplicity, brevity, and elegance.

Chapter II, "The second variation in the simplest class of problems," contains the Legendre and Jacobi criteria; the exposition, excellent already in the English edition, is even better in the present book and stands forth as a model of clearness and precision. Chapter III, "Sufficient conditions in the simplest class of problems," deals with the conditions for a weak minimum, the construction of a field of extremals, Weierstrass's expression for the second variation in terms of the E-function, which is here introduced by means of Hilbert's invariant integral, and various conditions for the existence of a strong minimum.

Chapter IV, "Auxiliary theorems on functions of a real variable," contains various lemmas on implicit functions and existence theorems for differential equations, preparatory to Chapter V, "Weierstrass's theory of the simplest class of problems in parametric representation," which treats anew,

* Reviewed by E. R. Hedrick in BULLETIN, vol. 12 (1906), pp. 80–90.

in parameter form, the topics dealt with in Chapters I–III besides bringing some new developments, chiefly Osgood's theorem. Chapter VI, "The case of variable end points," uses the parameter form and contains results due to Weierstrass, Kneser, and Bliss.

Chapter VII, "Kneser's theory," gives an exposition of Kneser's method of generalized geodetic coordinates, and Chapter VIII, "Discontinuous solutions," presents the results due to Weierstrass, Erdmann, and others.

Chapter IX, "The absolute extremum," proves the existence, according to Hilbert and Caratheodory, of an absolute minimum in cases where the function under the integral sign is positive for all possible values of dx/ds and dy/ds. Chapter X, "Isoperimetric problems," gives an exposition of the classical results, as well as those due to Weierstrass and Kneser, which are completed by the more recent researches of Lindeberg.

Thus far, the topics presented are all treated, although in a less complete fashion, in the English edition; the following chapters, however, are new. Of these, chapters XI, "The Euler-Lagrange multiplicator method," and XII, "Further necessary conditions, as well as sufficient ones, in the Lagrange problem," give a very complete and lucid presentation of the results hitherto obtained in the difficult problem of minimizing an integral containing n unknown functions subject to any number of accessory conditions.

Chapter XII, "Elements of the theory of the extrema of double integrals," gives a first introduction to this largely unexplored field, and the book closes with an index and an appendix containing an enumeration of the principal definitions and theorems, used throughout the book, concerning functions of a real variable.

The literature references are quite exhaustive, and the numerous exercises at the end of the chapters constitute, together with the illustrative examples given in the text, a practically complete collection of all special problems to be found in the literature since the days of Euler, thereby enhancing the value of this volume as a reference work.

To conclude: the book under review is a standard work of the highest merit, and will undoubtedly render still greater services to the investigators in this field than the English edition has already done.

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