Riemannschen Integrale erster Gattung" and "Algebraischer Beweis des Satzes von der Anzahl der linearunabhängigen Integrale erster Gattung," the titles of which convey a sufficient idea of the problems solved. However, it is a question whether Christoffel's later work was as novel and fundamental as that which we have discussed at greater length above.

At the end of certain memoirs the editor has added remarks which are helpful. In every way the books present a fine appearance, which may be a superfluous observation since they are published by Teubner.

LUTHER PFAHLER EISENHART.

PROBLEM COLLECTIONS IN CALCULUS.

Esercizi di Analisi infinitesimale. Di G. VIVANTI. Puntata I. Completo. Mattei, Pavia, 1912. ix + 470 pp. Price, 15 lire.

Sammlung von Aufgaben zur Anwendung der Differential- und Integralrechnung. Von F. DINGELDEY. (Lehrbücher der Mathematischen Wissenschaften $XXXII_{1,2}$.) I. Teil; Differentialrechnung, 1910, vi + 202 pp.; II. Teil; Integralrechnung, 1913, ii + 382 pp. Teubner, Berlin und Leipzig. Price, 6 + 13 marks.

MOST treatises on the calculus contain numerous solved and unsolved problems, but in what follows I wish to indicate some of the more notable separately published problem collections which, chiefly by reason of the industry of Germans, present a most formidable array before an inquirer. There are, on the one hand, works which simply contain problems to solve, as those of Byerly* and Wolstenholme.† On the other hand, we have the voluminous collection of books which give a synopsis of a certain amount of theory, set forth numerous worked out examples of a somewhat typical nature, and give similar problems for solution. This is the style of the little works by

* W. E. Byerly, Problems in Differential Calculus supplementary to a Treatise on Differential Calculus. Boston, 1895, pp. viii+71.

† J. Wolstenholme, Mathematical Problems on the Subjects for the Cambridge Math. Tripos Examination, Part I, 3d ed., Lond., 1891. Problems 1641–1992 on Differential Calculus, Higher Plane Curves, Integral Calculus.

Junker,* Pascal† and Dölp,‡ and of others of a more ambitious nature.

One of the earliest and most interesting among works of this latter kind, was compiled by George Peacock,‖ professor of mathematics in Cambridge University. Beside problems on the regulation topics of calculus courses, there is a section on higher plane curves and another of 180 pages on the integration of differential equations. Throughout are numerous historical notes. As Peacock had not the leisure to superintend the publication of a second edition of his "Examples," which, by 1841, had been long out of print, Gregory,§ a Fellow of Trinity, thought that he "should do a service to students by preparing a work on a similar plan, but with such modifications as seemed called for by the increased cultivation of analysis in the university." So he introduced "demonstrations of propositions which although important and interesting, do not usually find a place in works devoted to the exposition of the principles of the calculus." The choice of subjects was about the same as in Peacock and the exact references to, and quotations from, original sources form a valuable feature.

Drawing liberally upon Gregory's work, Frenet (famed for his formulas of differential geometry) published the pioneer collection¶ by a Frenchman. The third edition was enlarged to contain 657 problems, some of which dealt with double series, infinite products, Bernoulli's numbers, and symbolic operations, in addition to treatment, in the French manner, of the ordinary problems of English works. The sixth edition, published by Laurent after Frenet's death (1900), contained an

* Fr. Junker, Repetitorium und Aufgaben-Sammlung zur Differentialrechnung, 3. verb. Auflage, Leipzig, 1911, pp. 129. Repetitorium u. Auf. Sammlung z. Integralrechnung, 3. verb. Aufl., Leipzig, 1912, pp. 135 (Sammlung Göschen, 146, 147).

† E. Pascal, Esercizi critici di Calcolo differenziale e integrale (Manuali Hoepli), Milan, 1895, pp. xvi+275.

‡ Aufgaben zur Differential- und Integralrechnung, von H. Dölp. Neubearbeitet von E. Netto. 10. Aufl. Giessen, 1903, pp. 216. (First ed., 1869.) 12. Aufl., 1912, pp. xii+216, entitled "Grundzüge und Aufgaben, etc."

‖ Collection of Examples of the Applications of the Differential and Integral Calculus. Cambridge, 1820, pp. xiii+506+6 plates.

§ "Examples of the Processes of the Differential and Integral Calculus," collected by D. F. Gregory. Second edition edited [after Gregory's death in 1843] by Wm. Walton, Cambridge, 1846, pp. x+529+4 plates; French translation by Clarke, Paris, 1849.

¶J. F. Frenet, Recueil d'Exercices sur le Calcul infinitésimal. Paris, 1856; 3e éd., 1873; 5e éd. par H. Laurent, Paris, 1891; 6e éd. augmentée, Paris, 1904, pp. xlv+538. Russian ed., Moscow, 1899–1900.

appendix on residues, elliptic functions, partial differential equations, and total differential equations. Tisserand's elegant Recueil of 1877 consisted of solved problems only (differential calculus, 55; integral calculus, 41; applications of calculus to the solution of various questions concerning curves and surfaces, 70) and was intended to be supplementary to the third edition of Frenet. At the author's request Painlevé prepared the second edition,* and increased its size nearly 100 pages by adding a fourth part relative to the theory of functions, (27 exercises on definite integrals, residues, periodicity, inversion of integrals, critical points of various differential equations). Some problems were added to other parts. A Belgian work by Brahy,† more elementary and much less interesting, has gone through several editions.

Apart from the Bändchen of Pascal, Italy's contribution to this class of books is best represented by Vivanti's Esercizi. This is a work of considerable originality and of attractive get up. It contains some 600 worked out problems. About half of the book is taken up with the problems on the subjects of differentiation, limits, indeterminate forms, maxima and minima, integration, tracing and other discussion of some two score of curves. The latter half develops problems on osculating planes, curvature and torsion, differential equations.‡

Corresponding to the classic French treatise by Frenet, we have the German classic Uebungsbuch of O. Schloemilch.§ In the first edition, among features somewhat different from ordinary books of the kind, 40 pages are devoted to maxima and minima of a function, 52 to infinite series, 14 to functions and series of a complex variable. In the second part about 80 pages are given over to differential equations. Another

* F. Tisserand, Recueil complémentaire d'Exercices sur le Calcul infinitésimal. 2 éd. Augmentée de nouveaux Exercices par P. Painlevé; Paris, 1896, pp. xxiii+524.

† E. Brahy, Exercices méthodiques de Calcul différentiel, Brussels, 1867. 3e éd., Paris, 1905. Exercices méth. de Calcul intégral, Brussels, 1895; nouv. éd., Paris, 1903, pp. viii+301.

‡ When it is recalled that Vivanti is the author of the section "Infinitesimalrechnung," pp. 639–870, of Cantor's Vorlesungen über Geschichte der Mathematik, vol. 4 (1759–1799), Leipzig, 1908, the historical remarks in his Esercizi must be regarded as bearing the weight of trustworthy authority.

§ Uebungsbuch zum Studium der höheren Analysis, I. Theil: Differentialrechnung, Leipzig, 1868, pp. viii+264; 5. Aufl. von E. Naetsch, Leipzig, 1904, pp. viii+332. II. Theil: Integralrechnung, Leipzig, 1870, pp. viii+338; 4. Aufl. von R. Henke, 1900, pp. viii+418.

favorite work of a similar character is that by L. A. Sohncke,* who died (1853) shortly after the first edition was published.

All of the above works cover the same general field of pure mathematics. When we explore particular regions, we find, for example, the important books of P. Frost† and L. I. Magnus‡ in differential and analytical geometry and of R. d'Adhémar§ in theory of functions, quadratures, and differential equations.

There are also works in which the authors turn from pure mathematics, to dwell upon the applications of calculus to problems in such fields as mechanics and natural science—applications to so-called "practical problems." Many of the examples in Professor Byerly's little book (l. c.) are of this nature, but more recently, in America, we have "Applications of the Calculus to Mechanics," by Professors E. R. Hedrick and O. D. Kellogg.‖ Theory and definitions necessary for solving nearly 300 problems in vectors, statics, dynamics of a particle, work and energy, mechanics of a rigid body, are here given in clearly formulated fashion. And on six pages near the end are "Suggestions and Answers."

But Americans were not the first to publish such problems. More than 75 years ago D. C. L. Lehmus issued his book on mechanics and ballistics¶ and in later days we have the elaborate but unfinished "Anwendungen" of Arwed Fuhrmann** (died 1907). From these volumes the mathematician can glean many a fascinating "application" and the very carefully prepared and full bibliographies are a veritable mine of in-

* Sammlung von Aufgaben aus der Differential- und Integralrechnung. Halle, 1850; 5. Aufl. hrsg. von H. Amstein, 2 Bde. Halle, 1885; 6. Aufl. Bd. I: Differentialrechnung bearb. v. M. Lindow, Halle, 1903, pp. xi+304; Bd. II: Integralrechnung, hrsg. v. M. Lindow, Jena, 1905, pp. xii+221+vi+224.

† Hints for the Solution of Problems in the third edition of Solid Geometry, London, 1887, pp. 115.

‡ Sammlung von Aufgaben und Lehrsätzen aus der analytischen Geometrie. I. Theil, pp. 325–659. Berlin, 1833.

§ Exercices et Leçons d'Analyse. Paris, 1908, pp. viii+208.

‖ Boston, 1909, pp. vi+116.

¶ Anwendungen des höheren Kalküls auf geometrische, mechanische, insbesondere auf ballistische Aufgaben. Leipzig, 1836.

** Anwendungen der Infinitesimalrechnung in den Naturwissenschaften im Hochbau und in der Technik. Lehrbuch und Aufgabensammlung. In sechs Theilen, von denen jeder ein selbständiges Ganzes bildet. Theil I: Naturwissenschaftliche Anwendungen der Differentialrechnung. Berlin, 1888, pp. xii+148; Theil II: Naturwissenschaftliche Anwendungen der Integralrechnung, Berlin, 1890, pp. xii+268; Theil III: Bauwissenschaftliche Anwendungen der Differentialrechnung, Berlin, 1899, pp. xvi+348; Theil IV: Bauwiss. Anw. d. Integralrechnung, Berlin, 1903, pp. xiv+292.

formation for the inquirer. It may be observed that, at times, the treatment is intentionally more in the manner of a text-book on the subject in question, than of singly stated problems, the solutions of which involve calculus.

In concluding this general survey, reference may be given to four other works which are not primarily "problem collections," but are nevertheless of great interest for the large number of "practical" examples which they give. These works are J. W. Mellor's "Higher Mathematics for Students of Chemistry and Physics with special reference to Practical Work,"* W. Nernst and A. Schönflies's "Einführung in die mathematische Behandlung der Naturwissenschaft, kurzgefasstes Lehrbuch der Differential- und Integralrechnung mit besonderer Berücksichtigung der Chemie,"† John Perry's "Calculus for Engineers"‡ and Sir George Greenhill's "Notes on Dynamics."§

Let us now turn to the second of our books under review. Dingeldey has remarked that while Fuhrmann gave many problems in "Naturwissenschaften und Technik," the text-book character was also in evidence; that "applications" in "das ausgezeichnete Werk" of Perry were, primarily, applications of calculus to Technik; that Mellor's examples were taken almost wholly from Naturwissenschaften. He therefore felt that there was a lack of a book "das nur Aufgabensammlung sein will und Beispiele enthält, die der Geometrie, den Naturwissenschaften und der Technik entnommen sind." The volumes before us are the result of an endeavor to fill this want.

The author has taken pains "solche geometrische Beispiele zu bringen, die zumeist auch für den Physiker und Techniker von Interesse und Nützen sein werden, während der Studie-

* London, 1902, pp. xxi+543; 4 ed., 1913, pp. xxi + 641. Ger. ed. "in freier Bearbeitung," Berlin, 1906, pp. xi+412.

† München u. Berlin, 1895; 3. Aufl., 1901, pp. xii+340; 6. Aufl., 1910, pp. xii+442; 7 Auflage, 1913, pp. xii + 444. Russian ed., Moscow, 1907.

‡ London, 1897; eleventh impression, 1913, pp. vii+382. Ger. ed. Höhere Analysis für Ingenieure deutsch bearb. v. R. Fricke u. Fr. Süchting, Leipzig and Berlin, 1902, pp. ix+423; 2. Aufl. 1910, pp. xi+464. Russian ed., St. Petersburg, 1904.

§ Second edition, published by His Majesty's Stationery Office, London, 1908, pp. 222+15 plates, in folio format. This remarkable book, which may be purchased for the absurdly small sum of 3 shillings, can serve as an invaluable source of suggestion for arousing interest in courses on mechanics or calculus. I fancy that the "Hiawatha" and "Milton" problems first appeared in the 1893 edition of this work.

rende der reinen Mathematik hoffentlich den physikalischen und technischen Beispielen Interesse entgegenbringen wird, zumal diese so gewählt sind, dass ihr Verständnis keine besonderen Vorkentnisse erfordert; wo solche wünchenswert zu sein schienen, sind die nötigen Erläuterungen gegeben."

The work contains about 1000 Aufgaben with solutions and Beispiele accompanied by adequate explanation. At the beginning of each of the 43 sections are propositions and rules connected with the theme of the paragraph; no proofs are given. In contrast to several collections mentioned above, there are no paragraphs on differential equations.

Beispiele in Differential Calculus are not easy to find before the discussion of problems on maxima and minima. Nevertheless in connection with sections on Differentiation of exponential functions and of logarithms, on Functions of several variables, and on Introduction of a new variable, we have a discussion of Snellius's law of refraction, logarithmic and newtonian potential, Wronskian and functional determinants, Schwarzian differential invariant. Pedal curves (5 pages) and Series (24 pages) are subjects of two more sections. Discussion of points of inflexion leads to comment on the Hessian curve, and among the applications of maxima and minima we find the famous problem of the bee's cell, a rainbow problem, and questions in least squares. Anallagmatic curves and catacaustics come up naturally in the section on envelopes.

Almost at the beginning of the integral calculus volume and again, later on, are examples of Simpson's rule (with due credit to James Gregory) and its use in ship-building. As illustration of integrating trigonometric and cyclometric functions, Hooke's law and pressure of wind on a tower and on the sail of a boat are considered. Radio-activity is a Beispiel in the integration of exponential functions. Problems on Lagrange's interpolation formula, a falling body, velocity of chemical reactions, and on the Staukurve* enter into the section on integration of rational functions. In other sections we have: Fourier series; beta and gamma functions; elliptic integrals with pendulum problem; laws of the mean with applications to heat, machinery, hydraulics; rectification with

* I do not know an English equivalent for the name of this curve, to which ten pages are devoted by Dingeldey. It may be described as follows: suppose the water of a dammed stream to flow through a break in the dam; the form of the surface of the water above the dam in longitudinal profile is that of the Staukurve.

examples on catenary and telegraph wire, Fagnano's formula, parallel curves, spherical loxodrome, and Mercator's projection. The volume closes with sections on the volume of certain solids; determination of masses and of centers of gravity;* moments of inertia; applications of potential theory. But these brief indications of the contents of the volumes must suffice.

The author has accomplished the task set himself in admirable fashion. Carefully exact statements, in historical notes, in references to original sources, in indexes, and elsewhere are noticeable throughout. No especially serious error has come to the reviewer's notice. Problem 39 (I, 167) might well cause trouble since about a dozen signs are wrong and a ψ' should be substituted for a ϕ'. Most of the well-known curves are badly drawn. It would be an improvement if the indexes included all proper names mentioned in the volumes.

A few additions and notes at different points might increase the interest of the reader. For example: (1) after showing that a cardioid is a catacaustic of a circle (I, 183), reference might be given to Jacob Bernoulli (*Acta Eruditorum*, June, 1692, pages 293–295). (2) Heron of Alexandria may be credited with the solution of the problem of plane geometry: Given two points on the same side of a line, to find on the line a point such that the sum of its distances to the two given points shall be a minimum (I, 114). (3) The expression for the area of the maximum quadrilateral with given sides a, b, c, d (I, 146) might be given in the form $\sqrt{s(s-a)(s-b)(s-c)(s-d)}$ (where $2s = a+b+c+d$), with the remark that this was the expression for the area of an inscribed quadrilateral, given by Brahmagupta in the seventh century. The maximum property was shown by Lhuilier.† (4) When the length of the semicubical parabola (Neilsche Parabel, II, 271) has been found, why not remark that this was one of the first curves whose lengths were determined mathematically? The honor of effecting this determination belongs to Fermat and to the Englishman, William Neil. (5) After considering (II, 227) the integral $\int_0^\infty e^{-x^2}dx$ (which plays an important rôle in the theory of errors

* Pappus's theorems on the surface and volume of a solid of rotation, are introduced at this point. The usual mistake of attributing them to Guldin, who likely stole them from Pappus, is not committed here.

† Cf. Maximis et Minimis, Varsauiae, 1782, p. 18 ff.

in observation), make the transformation $x = \pm \frac{s}{\sqrt{2}}(1+i)$, where $i = \sqrt{-1}$, and it is readily found that $\frac{1}{\sqrt{2}}\int_0^\infty e^{-x^2}dx = \int_0^\infty \sin s^2\, ds = \int_0^\infty \cos s^2\, ds = \pm \frac{1}{2}\sqrt{\frac{\pi}{2}}$, integrals studied by Euler as early as 1781.* Now if we consider Jacob Bernoulli's problem,† to determine the curve whose curvature is proportional to its arc, we are led (on taking the constant of proportion as unity) to the equations

$$x = \int_0^s \sin s^2\, ds, \quad y = \int_0^s \cos s^2\, ds$$

which define a double spiral curve,‡ turning about the asymptotic points, determined by the Euler integrals above, and hence named by Cesàro the Clothoïde.§ It would also be interesting to remark that the curve is associated with the name of Fresnel, who was led to it in discussing the diffraction of light.‖

R. C. ARCHIBALD.

BROWN UNIVERSITY,
December, 1913.

SHORTER NOTICES.

Archimedis Opera Omnia. Volume II. By J. L. HEIBERG. Leipzig, B. G. Teubner, 1912. xviii + 554 pp. 8 Marks.

IT may seem strange that a new Latin-Greek edition of the works of Archimedes should be deemed necessary, the first one under the editorship of Professor Heiberg having appeared as late as 1880–1881. We expect new translations into modern

* "De valoribus integralium variabilis $x = 0$ usque $x = \infty$ extensorum." "M. S. Academiae exhib. d. 30 Aprilis, 1781." Euler here evaluates the integrals by means of gamma functions. Published in Inst. Calculi Integr. IV (1794), pp. 339–345.

† "Invenire curvam cujus curvedo in singulis punctis est proportionalis longitudini arcus; id est, quae ab appenso pondere flectitur in rectam," Opera, Geneva, 1774, vol. 2, pp. 1084–1086.

‡ Cf. Picard, Traité d'Analyse, tome 1, 2e ed., 1901, p. 357.

§ *Nouv. Ann. Math.* (3), vol. 5 (1886), p. 512.

‖ Œuvres complètes, tome I, p. 319, Paris, 1866; "Mémoire sur la diffraction de la lumière," presented to the Academy of Sciences in 1818, crowned in 1819 and first published in 1826. *Mém. de l'Acad. Fran.*, tome V, for 1821–22, Paris, 1826.