THE TWENTY-FIFTH REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-fifth regular meeting of the San Francisco Section of the Society was held at the University of Washington on May 22, 1914. Thirty-two persons were present, including the following members of the Society:

Professor R. E. Allardice, Professor W. A. Bratton, Dr. Thomas Buck, Professor A. F. Carpenter, Professor E. E. De Cou, Professor G. I. Gavett, Professor F. L. Griffin, Professor F. W. Hanawalt, Professor H. B. Leonard, Professor W. A. Manning, Professor R. E. Moritz, Professor F. M. Morrison, Dr. L. I. Neikirk, Dr. L. L. Smail, Professor W. M. Smith, Professor R. M. Winger.

Professor Manning, chairman of the section, presided. The members present lunched together between sessions at the Faculty Club, and were entertained in the evening at the home of Professor Moritz.

The following papers were presented at this meeting:

(1) Professor W. M. SMITH: "An associated curve."

(2) Dr. L. I. NEIKIRK: "The functional variable."

(3) Professor F. L. GRIFFIN: "Note on a measure of the variability of percentages in unequal samples recently proposed by H. B. Frost."

(4) Professor F. L. GRIFFIN: "An experiment in mathematical pedagogy."

(5) Dr. L. E. WEAR: "Self-dual rational quartics."

(6) Professor F. M. MORRISON: "On the relation between some important notions of projective and metric differential geometry."

(7) Professor A. F. CARPENTER: "Ruled surfaces with plane flecnode curves" (preliminary report).

(8) Dr. E. T. BELL: "Tables for the multiplication of four double theta functions."

(9) Dr. E. T. BELL: "An arithmetical theory of certain numerical functions."

(10) Dr. L. L. SMAIL: "Some theorems on triple limits."

(11) Dr. L. L. SMAIL: "A new general method for the summation of divergent series."

(12) Professor R. E. MORITZ: "On the general theory of cyclic-harmonic curves."

(13) Professor R. E. MORITZ: "The cyclo-harmonograph: a new mechanism for tracing curves whose equation is $\rho = a \cos p\theta/q + k$."

(14) Professor R. M. WINGER: "The sextic invariant of the Hesse group" (preliminary report).

Dr. Wear was introduced by Professor Moritz, and Dr. Bell by Dr. Neikirk. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

2. The study of integral equations has in recent years led to the development of a large body of theory called functional calculus.* One part of this is the theory of the functions of a line. Volterra† calls any quantity which depends for its value on the arc of a curve as a whole a function of the line.‡ He defines the partial derivative of the function at a point and derives a formula for the differential (variation). He extends the results to functions of more than one line. Following Volterra there have been several papers published along similar lines. In the present paper Dr. Neikirk gives a new definition of the function of a functional variable and also a new definition of the partial derivative at a point. He gets from this definition a mean value theorem and a Taylor's expansion. The mean value theorem gives, among other things, Euler's differential equation as a necessary condition in the calculus of variations. An example of the derivative is the following: the partial derivative of the length of the curve at a point is the negative of the curvature at the point, except for the end points, where it is infinite. A number of other results are given in the paper.

3. In studying the variation of percentages in a series of given samples, it is customary to compare the actual standard deviation for the given series with a theoretical limiting value, viz., $\sqrt{pq/h}$, where p = theoretical percentage for an indefinitely large number of cases, $q = 1 - p$, and h = harmonic mean size of the samples. Mr. Frost proposes to weight each sample with the number of individuals it contains,

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* See Winter, "Les principes du calcul fonctionnel," *Revue de Métaphysique et de Morale*, July, 1913.

† *Rendiconti della R. Accad. Dei Lincei*, vol. 3.

‡ See Hadamard, Leçons sur le Calcul des Variations, pp. 281–303.

and thus find a sort of generalized standard deviation, whose theoretical limiting value is $\sqrt{pq/\bar{n}}$, where $\bar{n}$ = arithmetical mean size of sample. Professor Griffin's note merely proves the mathematical validity of the process. It will form part of a paper intended for joint publication in the *American Naturalist.*

4. In this paper Professor Griffin describes a course taught for the past three years under the title of "An introduction to mathematical analysis," which has been evolved to meet these generally recognized needs: (1) for an earlier introduction of calculus, especially integral calculus; (2) for a closer correlation of the calculus with the preliminary subjects, and a consequent postponement of unessential parts of those subjects; and (3) for a fuller exposition of the actual uses of each topic at the time it is studied. A detailed outline is presented, showing how it is possible for students to acquire in a single year a considerable familiarity with graphical methods, calculus, trigonometry, college algebra, and analytic geometry,—in fact, to use elementary integral calculus in their first semester with a fair degree of facility and understanding. As the course presupposes nothing beyond elementary algebra and geometry, its subject matter is within the reach of students in the fourth year of preparatory schools. Some possible modifications and gains in later courses are also described.

5. A necessary condition for the self-duality of a plane curve is that the class of the curve be equal to the order, i. e., $n = n(n - 1) - 2d - 3c$. For the quartic curve there are two solutions, viz.,

$$d = 1, \quad c = 2; \quad d = 4, \quad c = 0,$$

which correspond respectively to the limaçon and the degenerate case of two conics. Dr. Wear finds that the former is invariant under a G_4 consisting of identity, a reflexion, and two polarities. The latter are invariant under a G_8 consisting of four collineations and four correlations. By specializing the two conics, groups of orders 16 and 48 may be obtained.

6. In his papers on the "Projective differential geometry of curved surfaces," *Transactions,* volumes 8, 9, and 10,

Professor E. J. Wilczynski has shown that the projective differential geometry of a surface may be based upon the consideration of a completely integrable system of two linear partial differential equations of the second order. In his study of the properties of a surface in the neighborhood of any one of its points, he has systematically used a certain semicovariant tetrahedron of reference. If the asymptotic lines are taken as the parametric lines, the differential equations used by Professor Wilczynski take exactly the same form as the familiar Gauss equations, and we can express directly the coefficients of the differential equations in terms of the Christoffel symbols used in the metrical theory of surfaces. We are thus able to change from the homogenous coordinate system to any suitable cartesian system.

Professor Morrison develops the transformations from the homogenous system to the special cartesian system consisting of the lines of curvature tangents, and the normal to the surface, and makes a study of some of the metrical properties of certain geometrical configurations associated with a point on a surface which have been defined and studied from a projective point of view by Professor Wilczynski. A special study is made of the metrical properties of the osculating linear complexes of the asymptotic curves and the pencil of complexes formed from these two. By a specialization upon the two directrices appearing in this pencil certain special points on a surface and special types of surfaces are defined and a study is made of these.

7. In this paper, Professor Carpenter sets up, in invariant form, the conditions for plane flecnode curves, and applies these conditions to the case of anharmonic curves and in particular to the case of conics.

8. There are 121 cases of the formula of H. J. S. Smith for the multiplication of four double theta functions (*Proceedings of the London Mathematical Society*, volume 10, pages 87–91; also Krazer, Lehrbuch der Thetafunktionen, Chapter VII; § 10). In using the double thetas, it was found convenient to have a compact form from which all of the biquadratic and other relations could be read off; it is the purpose of Dr. Bell's tables to supply this. Table I is a square matrix of the integers 0 to 15 with proper signs; Table II is a square matrix

of the symbols (μ, μ', ν, ν') equivalent respectively to Hermite's $\binom{\mu', \nu'}{\mu, \nu}$. It is explained with the tables how any one of the 121 quadruple products may be found by a single reference to each of the tables. For the triple thetas it is shown that there is a corresponding scheme in three dimensions; to Table I corresponds a cube, three of whose faces meeting in 0 are identical with Table I.

9. In this paper Dr. Bell continues a paper on "Numerical functions" presented to the Society at the New York meeting, October, 1912. The concept of prime functions here introduced simplifies the theory, whose objects are (1) to exhibit in concise form the interrelations arising in arithmetic between functions $f(n), f_1(n), \cdots$ which exist only when n is a positive integer, and are such that $f(mn) = f(m)f(n)$ when, and in general only when, m, n are relatively prime, $f(\ )$ being any one of the functions; (2) to construct a theory of the properties of these functions that shall be isomorphic to the theory of numbers. By definition, $f(n)$ is a unit function if for $n > 1$, $f(n) = 0$, and for $n = 1$, $f(n) = 1$. If n be factored in all possible ways into r factors, $n_1, n_2, \cdots, n_r$, and the sum of all the corresponding products $f_1(n_1)f_2(n_2) \cdots f_r(n_r)$ be formed, the result being regarded as a function of n, say $f(n)$, then $f(n)$ is defined to be the ideal product of $f_1(n), f_2(n), \cdots, f_r(n)$ (note that in the ideal product the arguments each $= n$). Examples of $f(\ )$ functions in use are the totient, the sum and number of divisors of an integer, $\mu(n)$ (Mertens), $\lambda(n)$ (Dirichlet); in all about 30. It is shown (1) that these are members of a class of functions denumerably infinite in number, (2) that any of these ∞ functions may be resolved into its ideal prime function factors in one way only, a prime function being an $f(\ )$ whose only factors are $f(\ )$ and unit functions; (3) that there are ∞ unit functions; (4) that any theorem in arithmetic regarding divisibility has a unique correspondent in this theory, as e. g., the number of prime functions is infinite; quotients are unique, etc. The theory of ideal addition for $f(\ )$ functions is constructed and shown to be isomorphic to addition for numbers; as also the theories of residuation, congruences, etc. These are too lengthy for abstraction.

10. Dr. Smail's first paper gives conditions under which certain triple limits exist, and gives relations between simultaneous and repeated triple limits.

11. In his Columbia dissertation (1913) (presented to the Society, April 26, 1913), Dr. Smail gave four general methods for the summation of divergent series. In the present paper, he gives a general method of summation which includes three of the four methods just mentioned. The general properties of this method are studied, and applied to the various special methods included in this general method. The known methods of summation of Cesàro, Riesz, Le Roy, Borel, Euler's power series method, and a method related to the Cesàro-Riesz methods of Hardy and Chapman, all appear as special cases under the general method here discussed.

12. Any curve whose equation in polar coordinates is $\rho = a \cos p\theta/q + k$, or whose equation may be reduced to this form by a transformation of coordinates, may be generated by a composition of a simple harmonic with a uniform circular motion. For this reason Professor Moritz designates all such curves by the term cyclic-harmonic. This class of curves evidently embraces an infinite number of species, only a few of which have received any systematic consideration. The present paper is the first of several papers dealing with the general properties of the whole class of curves. The topics considered in the first paper are: (1) the cartesian equation of the various species, (2) the number of species having the degree n, (3) the number and location of the axes of symmetry, (4) the number of polar maxima and minima and their location, (5) the number and distribution of double points.

13. The mechanism exhibited by Professor Moritz combines a simple harmonic motion, $\rho = c \cos pt + k$, with a uniform circular motion, $\theta = qt$. A crank-arm of length a imparts simple harmonic motion to a tracing pencil or pen which by means of a sliding clamp may be adjusted to different values of k. The paper on which the curve is to be drawn is fastened to a revolving disk. A train of gears enables p and q to assume separately all integral values from 1 to 10. The mechanism accomplishes thus the continuous description of 63 different species of curves. For each species a and k may be given all values $|a| \not> 3$ inches, $|k| \not> 5$ inches, subject only to the restriction $|a| + |k| < 6$ inches.

14. The complete system of invariants of the Hesse ternary

collineation group G_{216} was given by Maschke, *Mathematische Annalen,* volume 33, page 317. Of the five fundamental forms one is a proper sextic curve. Naturally this curve is closely associated with the syzygetic period of cubic curves which the group leaves invariant. Professor Winger discusses the sextic by means of its group property and points out some interesting relations to the system of cubics.

THOMAS BUCK,
Secretary of the Section.

THE RATIO OF THE ARC TO THE CHORD OF AN ANALYTIC CURVE NEED NOT APPROACH UNITY.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, September 9, 1913.)

"IF P is a fixed point on a curve and Q is a point which approaches P along the curve, the limit of the ratio of the arc PQ to the chord PQ is unity." While this statement is frequently made without reservation, it is easy, as in most analogous statements, to construct exceptions in the domain of real functions: by making the curve sufficiently crinkly the limit may become say two, or any assigned number greater than unity.

The object of this note, however, is to point out the necessity for reservation even in the domain of (complex) analytic curves. The limit may then be *less* than unity. For example, in the imaginary parabola

$$y = ix + x^2$$

the value of the limit in question, at the origin, is *not one, but about* .94. The exact value is easily found to be $\frac{2}{3}\sqrt{2}$.

Of course all such exceptions will be imaginary. Thus for a real non-circular ellipse the limit is obviously unity at each of the ∞^1 real points; but of the ∞^2 imaginary points of the ellipse, there are four points at which the limit takes the value .94 +. These are the points at which the tangent is a minimal (or isotropic) line. Thus if we explore *all* the points of the ellipse,