From (1), (6) and (8) it follows that

$$J'_m = \frac{1}{m^2}\int_0^{\pi/2}\frac{\sin^4 mu}{\sin^3 u}du + \eta'\cdot\frac{1}{m^2}\int_0^{\pi/2}\frac{\sin^4 mu}{\sin u}du,$$

$$\tfrac{1}{6} < \eta' < \left(\frac{\pi}{2} - 1\right),$$

and consequently

$$J'_m = 2\sum_{\lambda=m}^{2m-1}\frac{1}{2\lambda+1} + O\left(\frac{\log m}{m^2}\right),$$

$$\lim_{m=\infty} J'_m = \lim_{m=\infty}\sum_{\lambda=1}^{2m-1}\frac{2}{2\lambda+1} = \int_0^1\frac{dx}{1+x} = \log 2.$$

Using this result in connection with (4), it is seen that

$$\lim_{m=\infty}\frac{4J'_m}{J_m} = \frac{12\log 2}{\pi} = 2.648 - ,$$

so that, using the numerical values of J, and J',, we may finally state the result

$$2.758 - = \frac{4J_4'}{J_4} > \frac{4J_5'}{J_5} > \frac{4J_6'}{J_6} > \cdots > 2.648 - .$$

PRINCETON UNIVERSITY,
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NOTE ON THE ROOTS OF ALGEBRAIC EQUATIONS.

BY PROFESSOR R. D. CARMICHAEL AND DR. T. E. MASON.

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1. LANDAU* has established certain interesting inequalities concerning the least root of a class of algebraic equations, having been led to these results by considerations connected with his remarkable extension and generalization of Picard's famous theorem to the effect that an entire function which fails to assume two values is a constant. These special in-

* *Annales de l'École Normale Supérieure* (3), vol. 24 (1907), pp. 179–201; *Vierteljahrsschrift der Naturf. Gesellschaft in Zürich*, vol. 51 (1906), pp. 252–318.

equalities have been generalized and made more precise by Allardice.* Finally, still more comprehensive results have been obtained by Fejér † by a method which leaves nothing to be desired in the way of simplicity and elegance. Fejér's result, which contains those of the other authors mentioned, may be stated as follows:

Let

$$a_0 + a_1 z^{\nu_1} + a_2 z^{\nu_2} + \cdots + a_k z^{\nu_k} = 0 \tag{1}$$

be an equation of $k+1$ *terms in which*

$$a_0 \neq 0, \; a_1 \neq 0, \; \nu_1 < \nu_2 < \cdots < \nu_k.$$

Let ζ *be a root of this equation of least absolute value. Then we have the inequalities*

$$|\zeta| \leqq \left[\frac{\nu_2 \, \nu_3 \cdots \nu_k}{(\nu_2 - \nu_1)(\nu_3 - \nu_1) \cdots (\nu_k - \nu_1)}\right]^{1/\nu_1} \left|\frac{a_0}{a_1}\right|^{1/\nu_1}, \tag{2}$$

$$|\zeta| \leqq \left[\frac{(\nu_1 + 1)(\nu_1 + 2) \cdots (\nu_1 + k - 1)}{1 \cdot 2 \cdots (k-1)}\right]^{1/\nu_1} \left|\frac{a_0}{a_1}\right|^{1/\nu_1}. \tag{3}$$

In case $\nu_1 = 1$, *we have*

$$|\zeta| \leqq k \left|\frac{a_0}{a_1}\right|; \tag{4}$$

and this value is actually attained by the roots of the equation

$$a_0\left(1 + \frac{a_1 z}{k a_0}\right)^k = 0.$$

For the case $\nu_1 = 1$ formula (2) was obtained by Allardice. For the case when $k = 2$ and $\nu_1 = 1$ formula (4) was established by Landau. Furthermore, for $k = 3$ and $\nu_1 = 1$ Landau proved that $|\zeta|$ is not greater than $5\frac{2}{3}a_0/a_1$, a result less precise than that of Fejér above.

The principal purpose of this note is to establish other results concerning the least roots of algebraic equations, especially of those of certain special forms. It will be seen that these results are in some cases more far-reaching than those of Fejér. An additional theorem of a related nature is also given in the final section.

* Bulletin, vol. 13 (1907), pp. 443–447.

† *Comptes rendus*, vol. 145 (1907), pp. 459–461.

It is convenient, first of all, to simplify equation (1) in the following way: Divide each member by a_0 and replace z by x, where

$$x = \left(\frac{a_1}{a_0}\right)^{1/\nu_1} z.$$

Then the equation takes the form

$$1 + x^{\nu_1} + \alpha_2 x^{\nu_2} + \alpha_3 x^{\nu_3} + \cdots + \alpha_k x^{\nu_k} = 0. \tag{5}$$

We shall confine our attention principally to the latter equation.

2. We shall consider first the case in which $\nu_1 = 1$ and $\nu_{i+1} = \nu_i + 1$ when $i > 1$, writing the equation in the form

$$1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_m x^m = 0 \quad (s \geqq 1). \tag{6}$$

Let $x_1, x_2, \cdots, x_m$ denote the roots of this equation. From the equation whose roots are the reciprocals of those of (6) we see at once that

$$\Sigma \frac{1}{x_1} = -1, \; \Sigma \frac{1}{x_1 x_2} = 0, \; \cdots, \; \Sigma \frac{1}{x_1 x_2 \cdots x_s} = 0,$$

where the summation in each case is for all terms of the type written. From these equations and the customary formulas for the sum of the roots of an equation in terms of the elementary symmetric functions of these roots* it follows readily that

$$\Sigma \frac{1}{x_1{}^s} = (-1)^s.$$

Therefore

$$\frac{1}{|x_1|^s} + \frac{1}{|x_2|^s} + \cdots + \frac{1}{|x_m|^s} \geqq 1.$$

Now if we suppose that x_1 is a root of (6) of least absolute value, we have

$$\frac{m}{|x_1|^s} \geqq 1;$$

whence

$$|x_1| \leqq \sqrt[s]{m}.$$

* See Bôcher's Higher Algebra, p. 244.

This result may be stated in the form of the following theorem:

I. *The equation*

$$1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_m x^m = 0 \quad (s \geqq 1),$$

has a root which is not greater in absolute value than $\sqrt[s]{m}$, *whatever values* $c_{s+1}, c_{s+2}, \cdots, c_m$ *may have.*

COROLLARY. *The equation*

$$(7) \qquad 1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_{2^s}x^{2^s} = 0 \quad (s \geqq 1),$$

has a root which is not greater than 2 *in absolute value.*

The result of this corollary for the special case when $c_{s+2} = c_{s+3} = \cdots = c_{2^s} = 0$ is given by Landau (loc. cit.). Here the equation reduces to a trinomial.

If we apply to equation (7) the Allardice-Fejér formula (2) for $\nu_1 = 1$ we conclude that this equation has a root not greater in absolute value than $2^s/s$. For s approaching infinity this value $2^s/s$ itself approaches infinity at a rapid rate. Hence, the circle in which, according to the Allardice-Fejér result, equation (7) certainly has a root increases in size with increasing s. But from the theorem of our corollary it is seen that (7) always has a root in the circle of radius 2 about the point zero. Thus for the special case of equation (7) (and likewise of (6), as one may readily show) our results reach further than those of Allardice and Fejér.

We shall now show inversely that there is in general but one root of equation (6) which is bounded by the circle about zero of radius $\sqrt[s]{m}$ when $c_{s+1}, \cdots, c_m$ are arbitrary. More precisely, we shall prove the following theorem:

II. *Let* s *and* m *be any two positive integers having a prime number* p *between them*: $s < p < m$. *Let* M *be any positive constant. Then there exist equations of the form*

$$1 + x + c_{s+1}x^{s+1} + \cdots + c_m x^m = 0$$

having every root but one greater than M *in absolute value.*

To prove this theorem it is sufficient to construct equations having the specified property. Let $kp(k \geqq 1)$ be the greatest multiple of p which is less than m. We shall further specialize the equation to be constructed so that it shall have the form

$$(8) \qquad 1 + x + c_p x^p + c_{p+1}x^{p+1} + \cdots + c_{kp+1}x^{kp+1} = 0.$$

Now let ω be any primitive pth root of unity; then the set of pth roots of unity is $1, \omega, \omega^2, \cdots, \omega^{p-1}$. Let $x_1, x_2, \cdots, x_{kp+1}$ denote the $kp+1$ roots of equation (8). We shall build up equation (8) by properly choosing the values of its roots. Thus, we put

$$x_{kp+1} = -1,$$
$$x_{\mu k+\nu} = \omega^{-\mu}\tau, \quad (\mu = 0, 1, \cdots, p-1;\ \nu = 1, 2, \cdots, k),$$

where τ is any number whatever which is different from zero.

With these values of $x_1, x_2, \cdots, x_{kp+1}$ we have

$$\Sigma \frac{1}{x_1^t} = (-1)^t \quad (t = 1, 2, \cdots, p-1),$$

as one sees readily by means of the well-known relations

$$1^t + \omega^t + \omega^{2t} + \cdots + \omega^{(p-1)t} = 0 \quad (t = 1, 2, \cdots, p-1).$$

From the usual formulas (Bôcher, loc. cit.) for the sum of the roots of an equation in terms of the elementary symmetric functions of these roots it is now easy to see that

$$\Sigma\frac{1}{x_1} = -1, \quad \Sigma\frac{1}{x_1x_2} = 0, \quad \Sigma\frac{1}{x_1x_2x_3} = 0, \cdots, \quad \Sigma\frac{1}{x_1x_2\cdots x_{p-1}} = 0.$$

Hence, the equation of which the roots are $x_1, x_2, \cdots, x_{kp+1}$ has the form (8). If $|\tau| > M$, then all but one of the roots of the equation so formed are greater than M in absolute value. Hence we conclude to the theorem as stated above.

By means of II and the corollary to I we shall now prove the following theorem:

III. *Every equation of the form*

$$1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_{2^s}x^{2^s} = 0 \quad (s \geqq 1),$$

has at least one root not greater than 2 *in absolute value, while special equations of this form may have all roots but one greater in absolute value than any preassigned* M.

In order to complete the demonstration of this result it is sufficient to observe that obviously only one root is bounded when $s = 1$ and that a prime number p lies between s and 2^s when $s \geqq 2$. For $s = 2$ we have $p = 3$ and for $s = 3$ we have $p = 5$ or 7. For $s > 3$ we may prove the existence of p by means of Tschebyschef's theorem*: If an integer s is greater

* Bachmann, Niedere Zahlentheorie I, p. 66.

than 3 then there is at least one prime number between s and $2s - 2$.

3. Next, let us consider the equation

$$1 + x^r + c_{sr+1}x^{sr+1} + \cdots + c_m x^m = 0 \qquad (s \geqq 1). \tag{9}$$

Let $x_1, x_2, \cdots, x_m$ be the roots of this equation and denote by A_k and S_k respectively, the sums

$$A_k = \Sigma \frac{1}{x_1 x_2 \cdots x_k}, \quad S_k = \Sigma \frac{1}{x_1{}^k}.$$

Then we have*

$$\begin{aligned} S_m - A_1 S_{m-1} + A_2 S_{m-2} - \cdots & \\ + (-1)^{m-1} A_{m-1} S_1 + (-1)^m m A_m &= 0. \end{aligned} \tag{10}$$

Now from equation (9) we see readily that

$$A_1 = A_2 = \cdots = A_{r-1} = 0, \; A_r = (-1)^r, \; A_{r+1} = \cdots = A_{sr} = 0.$$

Hence it follows from (10) that

$$S_r = -r;$$

and then that

$$S_{2r} = r, \text{ if } s \geqq 2.$$

Continuing thus we have finally

$$S_{sr} = (-1)^s r;$$

that is,

$$\frac{1}{x_1{}^{sr}} + \frac{1}{x_2{}^{sr}} + \cdots + \frac{1}{x_m{}^{sr}} = (-1)^s r.$$

If x_1 is a root of (9) of least absolute value, then from the last equation it follows readily that

$$\frac{m}{|x_1|^{sr}} \geqq r \quad \text{or} \quad |x_1| \leqq \sqrt[sr]{\frac{m}{r}}.$$

Thus we have the following generalization of theorem I:

IV. *Every equation of the form* (9) *has at least one root which is less in absolute value than* $\sqrt[sr]{m/r}$.

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* See Bôcher's Higher Algebra, p. 244.

We may also generalize theorem II and so obtain the following result:

V. *Let r, s and m be any three positive integers such that r is prime while a prime number p lies between sr and $m - r + 1$; $sr < p < m - r + 1$. Let M be any positive constant. Then there exist equations of the form* (9) *which have all but r of their roots greater than M in absolute value.*

The detailed proof of this theorem will not be given. It is sufficient to construct an equation of the form

$$1 + x^r + c_p x^p + c_{p+1} x^{p+1} + \cdots + c_{kp+r} x^{kp+r} = 0 \quad (kp + r \leqq m),$$

by means of its roots $x_1, x_2, \cdots, x_{kp+r}$ defined as follows:

$$x_{\mu k+\nu} = \omega^{-\mu}\tau, \; (\mu = 0, 1, 2, \cdots, p-1; \; \nu = 1, 2, \cdots, k);$$

$$x_{kp+t} = \eta\epsilon^{-t} \quad (t = 1, 2, \cdots, r),$$

ω being a primitive pth root of unity, ϵ a primitive rth root of unity and η any rth root of -1.

4. Related to the foregoing theorems is the following rather obvious one:

VI. *In the equation*

$$(11) \qquad 1 + a_1 x + \cdots + a_k x^k + \cdots + a_m x^m = 0,$$

of degree m, let a_k be any given non-zero coefficient and let all the other coefficients be chosen in any manner whatever. The equation has at least one root which is not greater in absolute value than

$$(12) \qquad \sqrt[k]{\frac{m!}{k!(m-k)!}} \cdot \sqrt[k]{\frac{1}{|a_k|}}.$$

The equation

$$\left(1 + x\sqrt[k]{\frac{k!(m-k)!a_k}{m!}}\right)^m = 1 + a_1 x + \cdots + a_k x^k + \cdots + a_m x^m = 0$$

has all its roots equal in absolute value to the quantity (12).

Let $x_1, x_2, \cdots, x_m$ denote the roots of the given equation and let x_1 be a root of least absolute value. We have readily

$$(13) \qquad \Sigma \frac{1}{x_1 x_2 \cdots x_k} = (-1)^k a_k;$$

whence

$$\frac{1}{|x_1|^k}\frac{m!}{k!(m-k)!} \geqq |a_k|,$$

since the number of terms in the first member of (13) is obviously equal to the number of combinations of m things taken k at a time. Hence,

$$|x_1| \leqq \sqrt[k]{\frac{m!}{k!(m-k)!}}\sqrt[k]{\frac{1}{|a_k|}}.$$

This proves the first part of the theorem. The second part is obviously true.

5. We shall now prove a theorem of a nature somewhat different from that of those in the preceding sections; namely, the following:

VII. *All the roots of the equation*

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \tag{14}$$

are in absolute value less than or equal to

$$\sqrt{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}.$$

Let ζ be a root of least absolute value of the equation

$$1 + a_1z + a_2z^2 + \cdots + a_nz^n = 0.$$

Write

$$\frac{1}{1 + a_1z + a_2z^2 + \cdots + a_nz^n} = 1 + c_1z + c_2z^2 + \cdots. \tag{15}$$

Now, the circle of convergence of the power series in the second member of (15) passes through the point ζ. Its radius is therefore equal to $|\zeta|$. Therefore, by a well-known property of power series, we have

$$\frac{1}{|\zeta|} = \limsup_{m=\infty} \sqrt[m]{c_m}.$$

Hence, if $\bar{x}$ is a root of (14) of greatest absolute value, it is clear that

$$|x| = \limsup_{m=\infty} \sqrt[m]{c_m}. \tag{16}$$

Now, the value of c_m is readily expressed in the form of a determinant, as follows: If we multiply equation (15) through by the denominator of its first member and in the result equate coefficients of like powers of z, we have

$$\begin{aligned} c_1 \qquad\qquad\qquad\qquad &= -a_1, \\ a_1c_1 + c_2 \qquad\qquad\qquad &= -a_2, \\ a_2c_1 + a_1c_2 + c_3 \qquad\quad &= -a_3, \\ a_3c_1 + a_2c_2 + a_1c_3 + c_4 &= -a_4, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \quad \cdot . \end{aligned}$$

Hence

$$(17) \qquad c_m = - \begin{vmatrix} 1 & 0 & 0 & 0 \cdots 0 & a_1 \\ a_1 & 1 & 0 & 0 \cdots 0 & a_2 \\ a_2 & a_1 & 1 & 0 \cdots 0 & a_3 \\ a_3 & a_2 & a_1 & 1 \cdots 0 & a_4 \\ \cdot & \cdot & \cdot & \cdot \quad \cdot & \cdot \end{vmatrix} .$$

Now, if Δ_m is a determinant of order m,

$$\Delta_m = \begin{vmatrix} a_{11} & \cdots & a_{1m} \\ \cdot & \cdot \quad \cdot & \cdot \\ a_{m1} & \cdots & a_{mm} \end{vmatrix} ,$$

then

$$| \Delta_m | \leqq \sqrt{\sigma_1 \sigma_2 \cdots \sigma_m} ,$$

where

$$\sigma_i = \sum_{j=1}^{m} | a_{ij} |^2 .$$

This fundamental theorem is due to Hadamard.*

From this theorem and equation (17) we see readily that

$$(18) \qquad | c_m | < \{ 1 + | a_1 |^2 + | a_2 |^2 + \cdots + | a_n |^2 \}^{m/2},$$

since each row in the determinant in (17) has the property that the sum of the squares of the absolute values of its elements is not greater than $1 + | a_1 |^2 + \cdots + | a_n |^2$. From (18) and the result associated with (16) the theorem now follows immediately.

INDIANA UNIVERSITY,
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* *Bull. des Sciences Math.* (Darboux), vol. 17 (1893), pp. 240–246.