REMARKS ON FUNCTIONAL EQUATIONS.

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1. THE remarks of this article are in continuation of our paper in the BULLETIN, volume 19 (1912), pages 66–70. We have mentioned that a class of functional equations arises from the solution of the equation

$$\frac{\partial f(x, y)}{\partial x} + \mu(x, y)\frac{\partial f(x, y)}{\partial y} = 0, \tag{1}$$

where in particular we have assumed that $\mu(x, y) = \psi'(x)/\psi'(y)$. It is obvious that other classes of functional relations are obtained by considering the solution of the equation

$$\frac{dy}{dx} = \mu(x, y)$$

in connection with (1) when $\mu(x, y)$ is suitably specialized. The equation (1) may profitably be compared with another source of functional equations, that is, the equation*

$$\frac{\partial \varphi(x, y)}{\partial x} + \frac{\partial \varphi(x, y)}{\partial y} = \lambda(x, y). \tag{2}$$

The following functional relation is derivable from (1) and admits an explicit solution:†

$$\psi(x) - \psi(y) = \Omega^{-1}\{x\phi(y) - y\phi(x)\}. \tag{1'}$$

This equation can be solved in another manner by noticing that $\phi(x)$ can be formally eliminated. The result is

$$\Omega\{x - y\} = \Omega(x)\psi_1^{-1}(y) - \Omega(y)\psi_1^{-1}(x), \tag{1''}$$

where

$$\Omega(0) = 0, \quad \psi_1^{-1}(0) = 1,$$

$$\psi_1^{-1}(x) = \psi^{-1}\{x + \psi(1)\}.$$

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* Cf. V. Volterra, *Acc. Lincei*, vol. 19 (1910), pp. 169, 425; BULLETIN, vol. 19, p. 171; G. C. Evans, Proc. Cong. Math., Cambridge, 1913, vol. 1, p. 387; J. Hadamard, *L'Enseignement Mathématique*, vol. 14 (1912), p. 18, note; Paul Lévy, Paris thesis.

† Cf. BULLETIN, l. c., page 67.

We can now proceed as indicated by Abel, Œuvres, volume 1, page 10.

2. We consider the relations*

$$f(x, y) = \chi_f^{-1}\{\chi_f(x) - \chi_f(y)\}. \tag{3}$$

$$\phi(x, y) = \chi_f^{-1}\{\chi_f(x) + \chi_f(y)\}. \tag{4}$$

In (3) and (4) we write $\chi_f^{-1}(x)$ and $\chi_f^{-1}(y)$ for x and y respectively; then

$$f[\chi_f^{-1}(x), \chi_f^{-1}(y)] = \chi_f^{-1}\{x - y\}. \tag{3'}$$

$$\phi[\chi_f^{-1}(x), \chi_f^{-1}(y)] = \chi_f^{-1}\{x + y\}. \tag{4'}$$

Thus (3′) and (4′) represent respectively a subtraction formula and an addition formula.† It is thus seen that under differentiation, etc., symmetric and quasi-transitive‡ functional properties give rise to addition and subtraction formulas. In the philosophy of mathematics this result is of considerable interest. In symmetry and transitiveness and formal properties closely allied to these, many mathematical disciplines are rooted, e. g., the foundations of geometry and the theory of quaternions. The fact that certain properties of the indicated category induce addition and subtraction formulas tends to corroborate the fundamental character of the latter in the theory of functions.

3. We have shown§ purely formally that if

$$\phi\{f(y, z), z\} = y, \quad f\{\phi(y, z), y\} = z, \tag{5}$$

then ϕ can be eliminated; we obtained

$$f\{y, f(y, z)\} = z. \tag{5'}$$

In a similar manner f can be eliminated; for we have

$$\phi\{f[\phi(z, y), z], z\} = \phi(z, y),$$

that is,

$$\phi(y, z) = \phi(z, y). \tag{5''}$$

The relation (5″) may be used in the proof of theorem 1,

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* BULLETIN, l. c., p. 68.

† On this interpretation see Bourlet, *Ann. Sci. de l'Ecole Norm. Sup.*, vol. 14 (1897), pp. 140, 141. Independently of this reference, the preceding remark was kindly suggested to the author by Professor E. B. Van Vleck.

‡ BULLETIN, vol. 18 (1912), p. 300.

§ BULLETIN, vol. 19 (1912), p. 69.

BULLETIN, l. c., page 68, as expeditiously as relation (5′) in theorem 2, l. c., page 69. With regard to the duality, as it were, between the two theorems it is desirable to replace the hypothesis of theorem 1

$$\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}$$

by the relation

$$\phi\{y, \phi(x, z)\} = \phi\{z, \phi(x, y)\}.$$

The correspondence between (5′) and (5″) is possibly of use elsewhere in analysis.

From the equations

$$f\{y, f(x, z)\} = f\{z, f(x, y)\}, \tag{6}$$

$$\phi\{f(y, z), z\} = y, \quad f\{\phi(y, z), y\} = z,$$

we have shown l. c., page 69, that the ϕ may be formally eliminated; the result is

$$f\{f(x, z), f(x, y)\} = f(y, z). \tag{7}$$

We proceed to eliminate f from the equations (6). Substituting $\phi(x, z)$ for x in the first of equations (6) we get

$$f(y, x) = f\{z, f[\phi(x, z), y]\}.$$

Hence,

$$f\{\phi(y, z), x\} = f\{z, f[\phi(x, z), \phi(y, z)]\}.$$

Therefore*

$$f\{\phi[y, \phi(x, z)], x\} = \phi(y, z).$$

Substituting in this equation $f(x, z)$ for x we find

$$f\{\phi(y, x), f(x, z)\} = \phi(y, z). \tag{8}$$

Now

$$\phi\{f[\phi(y, x), f(x, z)], f(x, z)\} = \phi(y, x).$$

Therefore by (8),

$$\phi\{\phi(y, z), f(x, z)\} = \phi(y, x). \tag{9}$$

In (9) we write $\phi(z, x)$ for x; then

$$\phi\{\phi(y, z), x\} = \phi\{y, \phi(z, x)\}$$

or

$$\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}. \tag{10}$$

* From (6) we have $f\{f(y, z), f(x, z)\} = f(y, x)$ and consequently $(x, y) = f\{\varphi(x, z), \varphi(y, z)\}$.

Hence Abel's functional equations* result by eliminating f from equations (6).

In a similar manner we may eliminate ϕ from the equations

$$(11) \qquad \phi\{y, \phi(x, z)\} = \phi\{z, \phi(x, y)\},$$

$$\phi\{f(y, z), z\} = y, \quad f\{\phi(y, z), y\} = z.$$

To do this, we easily derive from equations (11) the equation (9); thence using the relation

$$f\{\phi[f(z, y), \phi(x, y)], \quad f(z, y)\} = \phi(x, y)$$

we obtain equation (8). From (9) there results in connection with (11)

$$(12) \qquad \phi\{f(x, y), z\} = \phi\{f(z, y), x\}.$$

And from (8) we derive

$$(13) \qquad f\{z, f(y, x)\} = \phi\{f(z, y), x\}.$$

From (12) and (13) we get finally the desired relation

$$(14) \qquad f\{x, f(y, z)\} = f\{z, f(y, x)\}.$$

The preceding formal eliminations enter into Grassmann's theory of synthetic and analytic associations which, as is well known, provides an abstract generalization of the processes of addition and subtraction.† In fact we may regard $\phi(x, y)$ and $f(x, y)$ as instances of Grassmann's associations $(x \cap y)$ and $(x \cup y)$ respectively; we have then in Grassmann's theory the theorems

I. $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$

imply

$(y \cap z) = (z \cap y)$ and $\{y \cup (y \cup z)\} = z.$

II. $\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}$

and $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$

imply

$\{y \cap (x \cap z)\} = \{z \cap (x \cap y)\}.$

III. $\{y \cap (x \cap z)\} = \{z \cap (x \cap y)\}$

and $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$

* Œuvres, vol. 1, p. 61.

† Cf. Gesammelte Werke, vol. 1, part 1, pp. 33–45.

imply

$$\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}.$$

IV. $$\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}$$

$$\text{and } \{y \cup (y \cup z)\} = z$$

imply

$$\{(x \cup y) \cup (x \cup z)\} = (z \cup y).$$

The study of the preceding relationship naturally would lead one to consider the monograph of Pietzker, Beiträge zur Funktionenlehre, Leipzig, 1899.*

4. The reference to Grassmann† in the preceding section suggests the following problem:

Given the quasi-transitive function $f(x, y)$ to find the quasi-transitive function $f_1(x, y)$ such that

$$f_1\{f(x, y), z\} = f\{f_1(x, z), f_1(y, z)\}. \tag{15}$$

We have

$$f(x, y) = \chi^{-1}\{\chi(x) - \chi(y)\}, \tag{16}$$

$$f_1(x, y) = \chi_1^{-1}\{\chi_1(x) - \chi_1(y)\}.$$

In (15) we put $z = z_0$, a constant; then if $\psi(x) = f_1(x, z_0)$ we have

$$\psi\{f(x, y)\} = f\{\psi(x), \psi(y)\}. \tag{15'}$$

Substituting in this relation for $f(x, y)$ from (16) we obtain

$$\chi\psi\chi^{-1}(x - y) = \chi\psi\chi^{-1}(x) - \chi\psi\chi^{-1}(y). \tag{17}$$

Hence

$$\psi(x) = \chi^{-1}\{c\chi(x)\}, \tag{18}$$

where c is an arbitrary constant. On the other hand, we put in relation (16)

$$\chi_2(x) = c'\chi_1(x);$$

then

$$f_1(x, z) = \chi_2^{-1}\{\chi_2(x) - c'\chi_1(z)\} \tag{16'}$$

or for $z = z_0$

$$\psi(x) = \chi_2^{-1}\{\chi_2(x) - c''\}, \tag{19}$$

where c'' is an arbitrary constant. As is well known, func-

* Cf. Moritz, *Amer. Journ. of Mathematics*, vol. 24, No. 3; A. F. Carpenter, same journal, vol. 35, No. 1.

† L. c., pp. 41–42.

tional relations of the types (18) and (19) have been treated by Schroeder and Abel respectively.* From (18) and (19) we have

$$\chi^{-1}\{c\chi(x)\} = \chi_2^{-1}\{\chi_2(x) - c''\}$$

or

$$\chi_2\chi^{-1}(cx) = \chi_2\chi^{-1}(x) - c''$$

or

$$(20)\qquad \chi_2\chi^{-1}(cx) = \chi_2\chi^{-1}(c) + \chi_2\chi^{-1}(x) - \chi_2\chi^{-1}(1).$$

The solution of (20) is

$$(21)\qquad \chi_2\chi^{-1}(x) = \log_a c_1 x,$$

where c_1 is a constant and the base a is arbitrary. From (21) we have

$$\chi_2(x) = \log_a [c_1\chi(x)].$$

Hence

$$(22)\qquad f_1(x, y) = \chi^{-1}\{c_1' \cdot \chi(x)/\chi(y)\}$$

which is the solution required.

Proceeding in a precisely analogous manner we can determine the symmetric function $\phi_1(x, y)$ such that

$$\phi_1\{\phi(x, y), z\} = \phi\{\phi_1(x, z), \phi_1(y, z)\},$$

where ϕ is a given symmetric function; we find

$$(23)\qquad \phi_1(x, y) = \chi^{-1}\{c_2\chi(x) \cdot \chi(y)\},$$

where c_2 is a constant and

$$\phi(x, y) = \chi^{-1}\{\chi(x) + \chi(y)\}.$$

A formal solution of a problem closely related to the above has been given by Pietzker, l. c., pages 9–10.

5. If we assume properties (6), (15) and the relation

$$(24)\qquad f_1\{\phi_1(y, z), y\} = z,$$

then there exists a function $\chi(x)$ such that

$$(25)\qquad \begin{aligned} \phi(x, y) &= \chi^{-1}\{\chi(x) + \chi(y)\}, \\ f(x, y) &= \chi^{-1}\{\chi(x) - \chi(y)\}, \\ f_1(x, y) &= \chi^{-1}\{\chi(x)/\chi(y)\}, \\ \phi_1(x, y) &= \chi^{-1}\{\chi(x) \cdot \chi(y)\}. \end{aligned}$$

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* Cf. M. J. van Uven, *Kon. Akad. van Wetenschappen te Amsterdam*, vol. 12 (1909–1910), pp. 208, 427; A. Hurwitz, *Math. Annalen*, vol. 70 (1910–1911), p. 33.

Inasmuch as certain formal properties of the real number system are satisfied* by the preceding functions, it seems appropriate to call $\phi(x, y)$ the *pseudo-sum* of x and y; the function $\phi_1(x, y)$, the *pseudo-product* of x and y, and similarly for the functions $f(x, y)$ and $f_1(x, y)$. Moreover, to zero corresponds $f(x, x)$, a constant, and to unity corresponds $f_1(x, x)$, a constant.

We can, of course, further condition the functions $\phi(x, y)$, $f(x, y)$, etc., by referring to other properties of the real number system. For example, by assuming the relation

$$f(tx, ty) = tf(x, y) \tag{26}$$

in addition to the relation (7) we find that there exists a function $\chi(x)$ such that†

$$\chi\{t \cdot \chi^{-1}(x - y)\} = \chi\{t \cdot \chi^{-1}(x)\} - \chi\{t \cdot \chi^{-1}(y)\}. \tag{27}$$

Hence

$$\chi\{t \cdot \chi^{-1}(x)\} = ax, \tag{28}$$

where a is an arbitrary constant. From (28) we have

$$\chi^{-1}(ax) \cdot \chi^{-1}(1) = \chi^{-1}(a)\chi^{-1}(x). \tag{29}$$

Therefore

$$\chi^{-1}(x) = \chi^{-1}(1) \cdot x^c,$$

where c is an arbitrary constant. Similarly, if instead of (26) we assume

$$f(tx, ty) = f(x, y)$$

or

$$f(t + x, t + y) = f(x, y),$$

we find respectively the relations

$$\chi^{-1}\{\chi(tx) - \chi(ty)\} = \chi^{-1}\{\chi(x) - \chi(y)\}$$

or

$$\chi^{-1}\{\chi(t + x) - \chi(t + y)\} = \chi^{-1}\{\chi(x) - \chi(y)\},$$

which yield,‡ essentially, well known functional relations due to Cauchy.

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* Cf. also Dickson, *Transactions*, vol. 4 (1903), pp. 14, 17.
† Cf. Bourlet's "additive transmutation," l. c., pp. 136, 137, 141, etc.
‡ Cf. Bourlet, l. c., p. 143 (8).