27. If a homogeneous fluid mass is rotating about an axis with a sufficiently small angular velocity there are two possible oblate spheroid figures of equilibrium, one nearly spherical, Σ_1, and one much flattened at the poles, Σ_2. If the fluid mass is compressible there are two figures of equilibrium nearly oblate spheroids. One approximates Σ_1 and is depressed in middle latitudes below the spheroid having the same polar and equatorial radii. The other approximates Σ_2 and is elevated in middle latitudes. The former theorem was proved by Airy, Callandreau and Darwin; the latter is a new theorem. This is established and a formula for the deviation from a true spheroid is obtained by Professor Moulton. He starts from an assumed relation between the pressure and density in the fluid and by making a compressibility parameter play a fundamental rôle is able to discuss the more flattened figure.

28. In his third paper Professor Carmichael shows how rational solutions of certain functional equations may be employed in solving problems of a certain class in the theory of diophantine analysis. In particular, several problems of Diophantus and Fermat are readily treated. The contents of both this paper and the preceding one by the same author will appear in his forthcoming "Introduction to Diophantine Analysis," to be published by Wiley and Sons.

H. E. SLAUGHT,
Secretary of the Chicago Section.

A GEOMETRIC DERIVATION OF A GENERAL FORMULA FOR THE SOUTHERLY DEVIATION OF FREELY FALLING BODIES.*

BY PROFESSOR WM. H. ROEVER.

(Read before the American Mathematical Society, October 25, 1913.)

WITHIN the last dozen years interest in the problem of the deviations of freely falling bodies seems to have been revived. There is a substantial agreement, among the writers who have treated this subject, as to the magnitude of the easterly deviation, their result being practically the same as that obtained

* See BULLETIN, vol. 20, No. 4, p. 175.

by Gauss more than one hundred years ago. Concerning the southerly deviation, however, there appears to be a diversity of results. This lack of agreement can, for the most part, be accounted for by the different definitions for southerly deviation and the different potential functions for the earth's field of force which have been used in the derivations of the various formulas. In another paper* the author shows that most of these formulas for the southerly deviation are special cases (corresponding to special potential functions and to special definitions) of his general formula

$$\text{(I)} \qquad \text{S.D.} = \left\{ 2\omega^2 \sin 2\phi_0 - 5\left(\frac{\partial g}{\partial \xi}\right)_0 \right\} \cdot \frac{h^2}{6g_0},$$

which may be written in the form

$$\text{(II)} \qquad \text{S.D.} = \xi_1 - \xi_2,$$

where

$$\xi_1 = \left\{ 2\omega^2 \sin 2\phi_0 + \left(\frac{\partial g}{\partial \xi}\right)_0 \right\} \cdot \frac{h^2}{6g_0}, \quad \xi_2 = \left(\frac{\partial g}{\partial \xi}\right)_0 \cdot \frac{h^2}{g_0},$$

S.D. representing the quantity which some writers use for southerly deviation and ξ_1 that which others use. Of the other quantities used in formula (I), h denotes the height through which the body falls, ω the angular velocity of the earth's rotation, ϕ_0, g_0, $(\partial g/\partial \xi)_0$ the values at the point P_0 from which the body falls of the astronomical latitude ϕ, the acceleration g due to weight, and the derivative of g with respect to ξ, where ξ represents distance measured to the south at P_0.

Formula (I), besides including as special cases many formulas which are based on particular potential functions, has a practical advantage over any of these formulas. This the following facts will show. In a country which is not very level the experimentally determined values of the derivative $(\partial g/\partial \xi)$ exceed, in magnitude, many times (sometimes ten or more) the values of this derivative which correspond to generally accepted potential functions. Furthermore, $(\partial g/\partial \xi)$ varies in an apparently very capricious manner. It may be positive at one station and negative at a station a few miles distant.† Therefore, formula (I), with the experimentally

* See the author's paper "Recent work on the deviations of falling bodies," soon to be published in *U. S. Monthly Weather Review*, vol. 43.

† See *Transactions Amer. Math. Society*, vol. 13, pp. 469–490 (Introduction); *Astronomical Journal*, nos. 670–672, Part II, § 4.

determined local value of the derivative $(\partial g/\partial \xi)$, is capable of accounting more accurately for the effect, on the southerly deviation of a falling body, of a local irregularity in the earth's field of force (such as a mountain) than any formula which is based on a particular (and consequently inadequate) potential function.

The author has already shown that, under the assumption of an asymmetric distribution of the earth's gravitating matter as well as under the assumption of a distribution of revolution, formula (I) is the first and predominating term of a still longer expression for the southerly deviation of freely falling bodies.* In view of the generality and practical nature of formula (I) it seems worth while to derive it by still another method. In this article a purely geometric derivation is given under the assumption of a distribution of revolution.

§ 1. *Definitions.*

Let us assume a set of rectangular axes $O - u, v, z$ (Fig. 1)

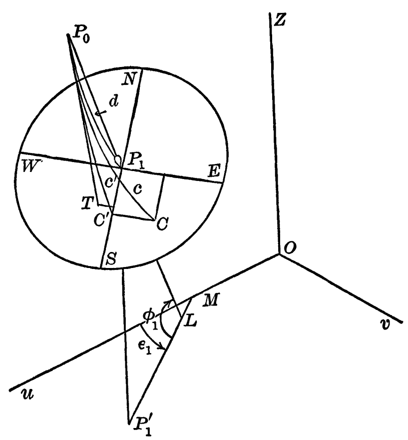

FIG. 1.

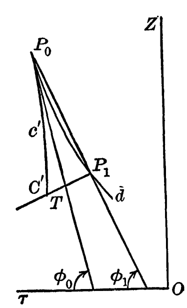

FIG. 2.

which are at rest with respect to the solid part of the earth and such that Oz coincides with the earth's axis of rotation, the positive sense of Oz being from the celestial south pole to the celestial north pole. The field of force which determines

* See *Transactions Amer. Math. Society*, vol. 12, pp. 335–353; vol. 13, pp. 469–490; and *Astronomical Journal*, nos. 670–672, Part II.

weight, i. e., the field in which the plumb-line is in equilibrium, is at rest with respect to these axes. We shall call this field of force the *weight field*. Let us denote by P_1 a point (on or near the earth's surface) which is at rest with respect to the axes $O - u, v, z$. The straight line which passes through P_1 and gives the direction of the force of the weight field at P_1 is defined as the *vertical of* P_1. The vertical of P_1 coincides with the string of a plumb-line, the bob of which is situated at P_1.* The *astronomical meridian plane of* P_1 is the plane which passes through the vertical of P_1 and is parallel to the axis of rotation Oz. The *astronomical latitude of* P_1 is the complement ϕ_1 (Fig. 1) of the angle which the vertical of P_1 (to the zenith) makes with the axis Oz (to the celestial north pole). The *astronomical longitude of* P_1 is the angle ϵ_1 (Fig. 1) which the meridian plane of P_1 makes with a fixed plane (zOu say) through Oz, and is measured from 0° to 360° to the east. The *horizontal plane of* P_1 is the plane which passes through P_1 and is perpendicular to the vertical of P_1. The *north-and-south line of* P_1 is the line of intersection of the meridian and horizontal planes of P_1. The *east-and-west line of* P_1 is the straight line which passes through P_1 and is perpendicular to the meridian plane of P_1. This line may also be defined as the intersection of the horizontal plane of P_1 by the plane which passes through P_1 and is perpendicular to the axis Oz.†

Now let us take a point P_0 (Fig. 1) above P_1 and in the vertical P_1P_0 of P_1. The vertical of P_0 will *not* coincide with the vertical of P_1, but it will pierce the horizontal plane $NESW$ of P_1 in a point T which does not coincide with P_1. Let us suppose a material particle to fall, under the influence of the earth's gravitation, from an initial position of rest (with respect to $O - u, v, z$) at P_0. The path of this particle, with respect to the axes $O - u, v, z$, is a curve c which pierces the horizontal plane $NESW$ of P_1 in a point C.

DEFINITION 1. *The perpendicular distance of the point C from the north-and-south line NP_1S of P_1 is called the easterly deviation of the falling particle, and is regarded as positive to the east* (see Fig. 1).

DEFINITION 2. *The perpendicular distance of the point C*

* It is assumed that the string of the plumb-line is weightless and perfectly flexible, and that the bob is a heavy particle.

† The above definitions are practically the same as those given by Pizzetti, Trattato di Geodesia teoretica, § 5 (1905).

from the east-and-west line EP_1W of P_1 is called the southerly deviation of the falling particle, and is regarded as positive to the south (see Fig. 1).

The following definitions will also be found convenient. *A line of force of the weight field* is a curve at each point P of which the tangent is the vertical. *A level (or equipotential) surface of the weight field* is a surface at each point P of which the normal is the vertical. The *plumb-bob locus of the point P_0* is the locus of the bobs of all plumb-lines (of the ideal type defined above) which are supported at P_0. Geometrically this locus may be defined in either of the two following ways:

(1) the locus of the points of tangency of tangents drawn from the fixed point P_0 to the lines of force of the weight field.

(2) the locus of the feet of perpendiculars dropped from the fixed point P_0 to the level surfaces of the weight field.

§ 2. *Some Fundamental Relations.*

Let us denote by ω the angular velocity (with respect to the inertial axes of the solar system) of the earth's rotation, and by V the potential function of the earth due to gravitational attraction alone, i. e., the function defined by the integral

$$V = \kappa \int \frac{dm}{\sqrt{(u-a)^2 + (v-b)^2 + (z-c)^2}},$$

where dm represents that element of mass which is situated at the point whose coördinates, with respect to the axes $O - u, v, z$, are a, b, c, the integration being extended throughout the whole volume of the earth, and κ is the gravitation constant. Then the potential function of the weight field is*

$$(1) \qquad W = V + \frac{\omega^2}{2}(u^2 + v^2).$$

Then the equations of the level surfaces of the weight field may be written in the form

$$W = K,$$

where K has constant values. The differential equations of the lines of force of the weight field are

$$(2) \qquad \frac{du}{W_u} = \frac{dv}{W_v} = \frac{dz}{W_z},$$

* See Pizzetti, loc. cit., § 2.

where W_u, W_v, W_z stand for the partial derivatives $\partial W/\partial u$, $\partial W/\partial v$, $\partial W/\partial z$ respectively. The equations of the plumb-bob locus of P_0 are

$$\frac{u-u_0}{W_u}=\frac{v-v_0}{W_v}=\frac{z-z_0}{W_z}. \tag{3}$$

From these equations it follows that P_0 (u_0, v_0, z_0) lies on this locus, and also that P_1 (u_1, v_1, z_1) lies on it, for since the straight line P_1P_0 is the vertical of P_1,

$$\frac{u_1-u_0}{W_u^{(1)}}=\frac{v_1-v_0}{W_v^{(1)}}=\frac{z_1-z_0}{W_z^{(1)}},$$

where $W_u^{(1)}$, $W_v^{(1)}$, $W_z^{(1)}$ denote the particular values of W_u, W_v, W_z respectively when $u=u_1$, $v=v_1$, $z=z_1$. It also follows from the systems of equations (2) and (3) that the plumb-bob locus of P_0 (represented by d in Fig. 1) has for tangent at P_0 the vertical P_0T of P_0.* Hence the

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* In order to prove this let us write equations (3) in the form

$$F=\begin{vmatrix} v-v_0 & z-z_0 \\ W_v & W_z \end{vmatrix}=0,\quad \Phi=\begin{vmatrix} z-z_0 & u-u_0 \\ W_z & W_u \end{vmatrix}=0.$$

Then for points of this curve

$$du:dv:dz=\begin{vmatrix} F_v & F_z \\ \Phi_v & \Phi_z \end{vmatrix}:\begin{vmatrix} F_z & F_u \\ \Phi_z & \Phi_u \end{vmatrix}:\begin{vmatrix} F_u & F_v \\ \Phi_u & \Phi_v \end{vmatrix}.$$

But

$$F_u=\begin{vmatrix} v-v_0 & z-z_0 \\ W_{vu} & W_{zu} \end{vmatrix},\quad F_v=\begin{vmatrix} v-v_0 & z-z_0 \\ W_{vv} & W_{zv} \end{vmatrix}+\begin{vmatrix} 1 & 0 \\ W_v & W_z \end{vmatrix},$$

$$F_z=\begin{vmatrix} v-v_0 & z-z_0 \\ W_{vz} & W_{zz} \end{vmatrix}+\begin{vmatrix} 0 & 1 \\ W_v & W_z \end{vmatrix},$$

$$\Phi_u=\begin{vmatrix} z-z_0 & u-u_0 \\ W_{zu} & W_{uu} \end{vmatrix}+\begin{vmatrix} 0 & 1 \\ W_z & W_u \end{vmatrix},\quad \Phi_v=\begin{vmatrix} z-z_0 & u-u_0 \\ W_{zv} & W_{uv} \end{vmatrix},$$

$$\Phi_z=\begin{vmatrix} z-z_0 & u-u_0 \\ W_{zz} & W_{uz} \end{vmatrix}+\begin{vmatrix} 1 & 0 \\ W_z & W_u \end{vmatrix}.$$

For the particular point P_0, (u_0, v_0, z_0) these derivatives have the values

$$F_u^{(0)}=0,\qquad F_v^{(0)}=W_z^{(0)},\quad F_z^{(0)}=-W_v^{(0)},$$

$$\Phi_u^{(0)}=-W_z^{(0)},\quad \Phi_v^{(0)}=0,\qquad \Phi_z^{(0)}=W_u^{0}.$$

Hence at P_0

$$du:dv:dz=W_z^{(0)}W_u^{(0)}:W_z^{0}W_v^{0}:W_z^{0}W_z^{0}=W_u^{0}:W_v^{0}:W_z^{0}.$$

It is assumed that the derivatives W_u, W_v, W_z do not all vanish at the same point (u, v, z) and that each of them has continuous first partial derivatives.

THEOREM 1. *The plumb-bob locus d of P_0 passes through the points P_0 and P_1 and is tangent at P_0 to the vertical P_0T of P_0* (see Fig. 1).

If we denote time by t, the differential equations of motion (with respect to the axes $O - u, v, z$) of the falling particle are*

$$(4)\qquad \begin{gathered} \frac{d^2u}{dt^2} - 2\omega\frac{dv}{dt} - \omega^2 u = \frac{\partial V}{\partial u}, \\ \frac{d^2v}{dt^2} + 2\omega\frac{du}{dt} - \omega^2 v = \frac{\partial V}{\partial v}, \quad \frac{d^2z}{dt^2} = \frac{\partial V}{\partial z}. \end{gathered}$$

Since by relation (1)

$$\omega^2 u + \frac{\partial V}{\partial u} = \frac{\partial W}{\partial u}, \quad \omega^2 v + \frac{\partial V}{\partial v} = \frac{\partial W}{\partial v}, \quad \frac{\partial V}{\partial z} = \frac{\partial W}{\partial z},$$

equations (4) may be written in the form:†

$$(5)\qquad \frac{d^2u}{dt^2} - 2\omega\frac{dv}{dt} = \frac{\partial W}{\partial u}, \quad \frac{d^2v}{dt^2} + 2\omega\frac{du}{dt} = \frac{\partial W}{\partial v}, \quad \frac{d^2z}{dt^2} = \frac{\partial W}{\partial z}.$$

The path c of the falling particle is that solution of equations (5) [or (4)] which is subject to the initial condition

$$(6)\qquad \begin{gathered} \text{when } t = 0, \quad u = u_0, \quad v = v_0, \quad z = z_0, \\ \frac{du}{dt} = \frac{dv}{dt} = \frac{dz}{dt} = 0. \end{gathered}$$

Therefore it follows from equations (5) that

$$(7)\qquad \left(\frac{d^2u}{dt^2}\right)_0 = W_u^{(0)}, \quad \left(\frac{d^2v}{dt^2}\right)_0 = W_v^{(0)}, \quad \left(\frac{d^2z}{dt^2}\right)_0 = W_z^{(0)},$$

where the subscript $_0$ indicates that those particular values of the derivatives d^2u/dt^2, d^2v/dt^2, d^2z/dt^2 have been taken for which $t = 0$. Since $W_u^{(0)}$, $W_v^{(0)}$, $W_z^{(0)}$ are proportional to the directional cosines of the vertical P_0T of P_0, we have, on account of relations (6) and (7), the

* For a derivation of these equations see § 2 of the author's paper in the *Transactions Amer. Math. Society*, vol. 13, pp. 469–490, where x, y, z, U stand for u, v, z, V here used.

† A superficial reading of equations (5) might lead one *erroneously* to the conclusion that ω^2 was neglected. In these equations ω^2 is contained in the expressions of the derivatives $\partial W/\partial u$, $\partial W/\partial v$. In short, equations (5) are the equivalents of equations (4), and *not* the approximations thereto made by neglecting ω^2.

THEOREM 2. *The path c of the falling particle* (*with respect to the axes $O - u, v, z$*) *passes through the point P_0 and is tangent at P_0 to the vertical P_0T of P_0* (see Fig. 1).

§ 3. *Special Relations Corresponding to a Distribution of Revolution.*

By a *distribution of revolution* of the earth's gravitating matter we shall mean a distribution of matter for which the potential function (1) assumes the form

$$W = f(\tau, z), \tag{8}$$

where

$$\tau = \sqrt{u^2 + v^2}$$

represents the distance of the point (u, v, z) from the earth's axis of rotation Oz.

For such a distribution the level surfaces of the weight field are surfaces of revolution, the two-parameter family of lines of force of the weight field lie in the one-parameter family of planes which pass through the axis Oz, and the differential equation of those lines of force which lie in any such plane is

$$\frac{dz}{d\tau} = \frac{\partial f / \partial z}{\partial f / \partial \tau}; \tag{9}$$

the plumb-bob locus of P_0 lies in the plane of P_0 and Oz, and its equation in this plane is

$$\frac{\partial f}{\partial z}(\tau - \tau_0) - \frac{\partial f}{\partial \tau}(z - z_0) = 0. \tag{10}$$

The meridian planes of P_1 and P_0 then coincide and contain, besides the verticals of P_1 and P_0, the plumb-bob locus d of P_0. The path c of the falling particle does not lie in this plane. Let us project it orthographically on this plane and denote by c' and C' the projections of c and C respectively (see Fig. 2). Then it follows from Definition 2, § 1, that

$$\text{S.D.} = P_1C', \tag{11}$$

where S.D. stands for southerly deviation. The point T in which the vertical of P_0 pierces the horizontal plane of P_1 now (i. e., for a distribution of revolution) lies in the north-and-south line of P_1. Let us put (see Fig. 2)

$$\xi_1 = TC', \quad \xi_2 = TP_1,$$

the positive sense of each of these quantities being the same as that of S.D. Then

(12) $\xi_1 - \xi_2 = TC' - TP_1 = P_1T + TC' = P_1C' =$ S.D.

The curve c' is closely related to a curve c'' which we will now define. The path c of the falling particle may be regarded as the directrix of a surface of revolution of axis Oz. This surface of revolution passes through the point P_0. The intersection of this surface of revolution by the plane of P_0 and Oz is the curve c''.

We will now prove the two following theorems.

THEOREM 3. *The curvature, at P_0, of the plumb-bob locus d of P_0 is twice that of the line of force of the weight field which passes through P_0, and the concavity of these curves is in the same direction.*

THEOREM 4. *The curvature, at P_0, of the curves c' and c'' is the same in both magnitude and direction.*

In order to prove Theorem 3 let us note that, since the two curves are tangent, it will be sufficient to compare the values, which correspond to the two curves, of the second derivative $d^2z/d\tau^2$. For the line of force the expression for this second derivative is found by differentiating equation (9). Thus we find for the line of force that

$$(13)\qquad \frac{d^2z}{d\tau^2} = \frac{\dfrac{\partial^2 f}{\partial\tau\partial z}\left\{\left(\dfrac{\partial f}{\partial\tau}\right)^2 - \left(\dfrac{\partial f}{\partial z}\right)^2\right\} + \dfrac{\partial f}{\partial\tau}\dfrac{\partial f}{\partial z}\left\{\dfrac{\partial^2 f}{\partial z^2} - \dfrac{\partial^2 f}{\partial\tau^2}\right\}}{\left(\dfrac{\partial f}{\partial\tau}\right)^3}.$$

Let us represent by $F(\tau, z)$ the left member of equation (10). The derivatives $dz/d\tau$, $d^2z/d\tau^2$ of the function defined by equation (10) are expressible in terms of the partial derivatives of F by means of the well known relations

$$(14)\qquad \begin{gathered} \frac{\partial F}{\partial\tau} + \frac{\partial F}{\partial z}\cdot\frac{dz}{d\tau} = 0, \\ \frac{\partial^2 F}{\partial\tau^2} + 2\frac{\partial^2 F}{\partial\tau\partial z}\cdot\frac{dz}{d\tau} + \frac{\partial^2 F}{\partial z^2}\left(\frac{dz}{d\tau}\right)^2 + \frac{\partial F}{\partial z}\cdot\frac{d^2z}{d\tau^2} = 0. \end{gathered}$$

For the particular point $P_0(\tau_0, z_0)$ we have*

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* Since $F = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \dfrac{\partial f}{\partial\tau} & \dfrac{\partial f}{\partial z} \end{vmatrix}$,

$$\left(\frac{\partial F}{\partial \tau}\right)_0 = \left(\frac{\partial f}{\partial z}\right)_0, \quad \left(\frac{\partial F}{\partial z}\right)_0 = -\left(\frac{\partial f}{\partial \tau}\right)_0,$$

$$(15) \quad \left(\frac{\partial^2 F}{\partial \tau^2}\right)_0 = 2\left(\frac{\partial^2 f}{\partial \tau \partial z}\right)_0, \quad \left(\frac{\partial^2 F}{\partial \tau \partial z}\right)_0 = \left(\frac{\partial^2 f}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f}{\partial \tau^2}\right)_0,$$

$$\left(\frac{\partial^2 F}{\partial z^2}\right)_0 = -2\left(\frac{\partial^2 f}{\partial \tau \partial z}\right)_0 .$$

Hence for the plumb-bob locus we have, at P_0,

$$\left(\frac{dz}{d\tau}\right)_0 = -\frac{\left(\frac{\partial F}{\partial \tau}\right)_0}{\left(\frac{\partial F}{\partial z}\right)_0} = \frac{\left(\frac{\partial f}{\partial z}\right)_0}{\left(\frac{\partial f}{\partial \tau}\right)_0},$$

$$(16) \quad \left(\frac{d^2 z}{d\tau^2}\right)_0 = \frac{\left(\frac{\partial^2 F}{\partial \tau^2}\right)_0 + 2\left(\frac{\partial^2 F}{\partial \tau \partial z}\right)_0\left(\frac{dz}{d\tau}\right)_0 + \left(\frac{\partial^2 F}{\partial z^2}\right)_0\left(\frac{dz}{d\tau}\right)_0^2}{-\left(\frac{\partial F}{\partial z}\right)_0}$$

$$= 2\,\frac{\left(\frac{\partial^2 f}{\partial \tau \partial z}\right)_0\left\{\left(\frac{\partial f}{\partial \tau}\right)_0^2 - \left(\frac{\partial f}{\partial z}\right)_0^2\right\} + \left(\frac{\partial f}{\partial \tau}\right)_0\left(\frac{\partial f}{\partial z}\right)_0\left\{\left(\frac{\partial^2 f}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f}{\partial \tau^2}\right)_0\right\}}{\left(\frac{\partial f}{\partial \tau}\right)_0^3}.$$

$$\frac{\partial F}{\partial \tau} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^2 f}{\partial \tau^2} & \frac{\partial^2 f}{\partial z \partial \tau} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ \frac{\partial f}{\partial \tau} & \frac{\partial f}{\partial z} \end{vmatrix},$$

$$\frac{\partial F}{\partial z} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^2 f}{\partial \tau \partial z} & \frac{\partial^2 f}{\partial z^2} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ \frac{\partial f}{\partial \tau} & \frac{\partial f}{\partial z} \end{vmatrix},$$

$$\frac{\partial^2 F}{\partial \tau^2} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^3 f}{\partial \tau^3} & \frac{\partial^3 f}{\partial z \partial \tau^2} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ \frac{\partial^2 f}{\partial \tau^2} & \frac{\partial^2 f}{\partial z \partial \tau} \end{vmatrix} + \frac{\partial^2 f}{\partial z \partial \tau},$$

$$\frac{\partial^2 F}{\partial \tau \partial z} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^3 f}{\partial^2 \tau \partial z} & \frac{\partial^3 f}{\partial z^2 \partial \tau} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ \frac{\partial^2 f}{\partial \tau^2} & \frac{\partial^2 f}{\partial z \partial \tau} \end{vmatrix} + \frac{\partial^2 f}{\partial z^2},$$

$$\frac{\partial^2 F}{\partial z^2} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^3 f}{\partial \tau \partial z^2} & \frac{\partial^3 f}{\partial z^3} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ \frac{\partial^2 f}{\partial \tau \partial z} & \frac{\partial^2 f}{\partial z^2} \end{vmatrix} - \frac{\partial^2 f}{\partial \tau \partial z}.$$

The first of these relations being the same as equation (9) for the point P_0 furnishes another proof of the fact, contained in Theorem 1, that the plumb-bob locus of P_0 is tangent, at P_0, to the line of force (of the weight field) which passes through P_0. This fact, taken in conjunction with the fact that the second of expressions (16) is $+\,2$ times as great as expression (13) for the point P_0, constitutes the proof of Theorem 3.

In order to prove Theorem 4 let us express the equations of motion of the falling particle in terms of cylindrical coordinates. For this purpose let us put

$$u = \tau \cos \lambda, \quad v = \tau \sin \lambda, \quad z = z, \tag{17}$$

where $\tau = \sqrt{u^2 + v^2}$ has the same meaning as above, and λ is the angle which the plane through Oz and the general point (u, v, z) makes with the plane zOu. If we subject the system of equations (4) to this transformation, we obtain the equations

$$\begin{gathered} \frac{d^2\tau}{dt^2} - \tau\left(\omega + \frac{d\lambda}{dt}\right)^2 = \frac{\partial V}{\partial \tau}, \quad \frac{d}{dt}\left\{\tau^2\left(\omega + \frac{d\lambda}{dt}\right)\right\} = \frac{\partial V}{\partial \lambda}, \\ \frac{d^2 z}{dt^2} = \frac{\partial V}{\partial z}. \end{gathered} \tag{18}$$

By the same transformation relation (1) becomes

$$W = V + \frac{\omega^2}{2}\tau^2. \tag{19}$$

If we assume the potential function to be of the form (8),

$$\frac{\partial W}{\partial \lambda} = \frac{\partial V}{\partial \lambda} = 0, \tag{20}$$

and then the second of equations (18) yields the integral

$$\frac{d\lambda}{dt} + \omega = \frac{k}{\tau^2}, \tag{21}$$

where k is the constant of integration. The first of equations (18) then becomes

$$\frac{d^2\tau}{dt^2} = \frac{\partial V}{\partial \tau} + \frac{k^2}{\tau^3} = \frac{\partial f}{\partial \tau} - \omega^2\tau + \frac{k^2}{\tau^3}. \tag{22}$$

Furthermore, when $t = 0$ we have $d\lambda/dt = 0$ and $\tau = \tau_0$, whence by (21)

$$k = \omega\tau_0^2. \tag{23}$$

If now we put

$$\Omega(\tau, z) = f(\tau, z) - \frac{\omega^2}{2}\left(\tau^2 + \frac{\tau_0^4}{\tau^2}\right), \tag{24}$$

the equations of motion (for a distribution of revolution) take the form

$$\frac{d^2\tau}{dt^2} = \frac{\partial\Omega}{\partial\tau}, \quad \frac{d^2z}{dt^2} = \frac{\partial\Omega}{\partial z}, \quad \frac{d\lambda}{dt} = \omega\left(\frac{\tau_0^2}{\tau^2} - 1\right). \tag{25}$$

The path c of the falling particle is (for a distribution of revolution) that solution of equations (25) which is subject to the initial conditions

$$\begin{aligned} &\text{when } t = 0, \quad \tau = \tau_0 = \sqrt{u_0^2 + v_0^2}, \qquad z = z_0, \\ &\lambda = \lambda_0 = \operatorname{arc\,tan}\frac{v_0}{u_0}, \quad \frac{dz}{dt} = \frac{d\tau}{dt} = \frac{d\lambda}{dt} = 0. \end{aligned} \tag{26}$$

The curve c'' is now easily seen to be that solution of the pair of differential equations

$$\frac{d^2\tau}{dt^2} = \frac{\partial\Omega}{\partial\tau}, \quad \frac{d^2z}{dt^2} = \frac{\partial\Omega}{\partial z} \tag{27}$$

which is subject to the initial conditions

$$\text{when } t = 0, \quad \tau = \tau_0, \quad z = z_0, \quad \frac{d\tau}{dt} = \frac{dz}{dt} = 0. \tag{28}$$

For the same ordinate z the abscissas of the two curves c'' and c' differ by the amount

$$\begin{aligned} &\tau - \tau\cos(\lambda - \lambda_0) \\ &\qquad = 2\tau\sin^2\tfrac{1}{2}(\lambda - \lambda_0) = \tau[\tfrac{1}{2}(\lambda - \lambda_0)^2 + \cdots], \end{aligned} \tag{29}$$

where z and τ are those solutions of equations (27) which are subject to the conditions (28), or where z, τ, λ are those solutions of equations (25) which are subject to the conditions (26).

The solutions τ, z to terms of order not higher than the second in t are easily seen to be

$$\tau = \tau_0 + \alpha_2 t^2 + \cdots, \quad z = z_0 + \beta_2 t^2 + \cdots, \tag{30}$$

where

$$\alpha_2 = \frac{1}{2}\left(\frac{\partial \Omega}{\partial \tau}\right)_0, \quad \beta_2 = \frac{1}{2}\left(\frac{\partial \Omega}{\partial z}\right)_0,$$

the subscript $_0$ indicating that the values of the derivatives to which it is attached correspond to $\tau = \tau_0$ and $z = z_0$. Hence

$$\frac{\tau_0^2}{\tau^2} = 1 - \frac{2\alpha_2}{\tau_0} t^2 + \cdots,$$

and therefore, by the last of equations (25),

$$\frac{d\lambda}{dt} = \omega\left(-\frac{2\alpha_2}{\tau_0} t^2 + \cdots\right), \tag{31}$$

whence, by (26),

$$\lambda - \lambda_0 = -\tfrac{2}{3}\omega\frac{\alpha_2}{\tau_0} t^3 + \cdots. \tag{32}$$

Therefore the difference (29) becomes

$$\begin{aligned} \tau - \tau\cos(\lambda - \lambda_0) &= (\tau_0 + \alpha_2 t^2 + \cdots)\left(\frac{2}{9}\frac{\alpha_2^2\omega^2}{\tau_0^2} t^6 + \cdots\right) \\ &= \frac{2}{9}\frac{\alpha_2^2\omega^2}{\tau_0} t^6 + \cdots \\ &= \frac{2}{9}\frac{\omega^2\alpha_2^2}{\tau_0\beta_2^3}(z - z_0)^3 + \cdots. \end{aligned} \tag{33}$$

Since this difference is of the third order in $(z - z_0)$ it follows that the curves c' and c'' have contact of the second order at the point P_0 and hence we have proved Theorem 4.

§ 4. *Some Known Theorems.*

The following three theorems will be found useful:

THEOREM 5. *In a two-dimensional field of force, for which a potential function exists, the curvature of a line of force is the derivative of the logarithm of the force taken in that direction, on an equipotential curve, in which the force increases.*

THEOREM 6. *If, in a positional field of force, a particle starts from rest, the initial curvature of the path described is one-third of the curvature of the line of force through the initial position.**

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* See Kasner's Princeton Colloquium Lectures, p. 9, footnote 2.

THEOREM 7. *If at a point P of a curve (with a continuously turning tangent) at which the curvature is finite (i. e., neither zero nor infinite) the tangent and the inner normal be taken as a set of rectangular axes, then the equation of the curve referred to these axes may be written in the form*

$$y = \frac{1}{2} \cdot \frac{1}{\rho} x^2 + \text{an expression of order three, or higher, in } x,$$

*where x is measured from P along the tangent, y is measured from P along the inner normal and ρ is the radius of curvature at P.**

References to proofs of Theorems 6 and 7 are given in the following footnotes. In order to prove Theorem 5 let us refer the field of force to a set of rectangular axes (x, y) and denote by X, Y the components of the force j of the field. Then the differential equation of the lines of force is

$$\frac{dy}{dx} = \frac{Y}{X}, \tag{34}$$

whence

$$\frac{d^2y}{dx^2} = \frac{X^2 \dfrac{\partial Y}{\partial x} - Y^2 \dfrac{\partial X}{\partial y} + XY\left(\dfrac{\partial Y}{\partial y} - \dfrac{\partial X}{\partial x}\right)}{X^3}. \tag{35}$$

On the other hand, the magnitude of the force is

$$j = \sqrt{X^2 + Y^2} = X \cos \phi + Y \sin \phi, \tag{36}$$

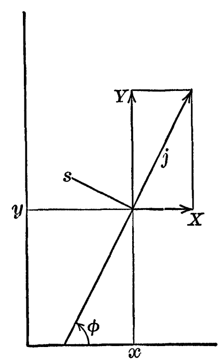

FIG. 3.

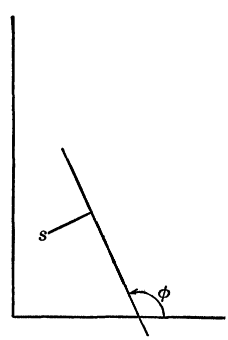

FIG. 4.

where $\tan \phi$ is the slope of the vector representing the force

* See Osgood's Introduction to Infinite Series, p. 39. Also Goursat-Hedrick, Mathematical Analysis, vol. 1, p. 451.

(Fig. 3). Hence

$$0 = -X \sin \phi + Y \cos \phi,$$

whence

$$j \cos \phi = X, \quad j \sin \phi = Y.$$

Then

$$\frac{\partial j}{\partial x} = \frac{1}{j}\left(X\frac{\partial X}{\partial x} + Y\frac{\partial Y}{\partial x}\right), \quad \frac{\partial j}{\partial y} = \frac{1}{j}\left(X\frac{\partial X}{\partial y} + Y\frac{\partial Y}{\partial y}\right).$$

If now we represent by s (Figs. 3 and 4) distance measured on a line perpendicular to the vector representing j and such that s is positive on the side opposite to that on which the angle ϕ lies, then

$$\frac{\partial j}{\partial s} = -\frac{\partial j}{\partial x}\sin\phi + \frac{\partial j}{\partial y}\cos\phi = \frac{1}{j}\left(-\frac{\partial j}{\partial x}Y + \frac{\partial j}{\partial y}X\right)$$

$$(37) \qquad = \frac{X^2\dfrac{\partial X}{\partial y} - Y^2\dfrac{\partial Y}{\partial x} + XY\left(\dfrac{\partial Y}{\partial y} - \dfrac{\partial X}{\partial x}\right)}{j^2}.$$

If now

$$(38) \qquad \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x},$$

which is the case when a potential function exists, then by (35) and (37),

$$(39) \qquad \frac{d^2y}{dx^2} = \frac{j^2}{X^3}\cdot\frac{\partial j}{\partial s},$$

whence

$$(40) \qquad \frac{\partial}{\partial s}\log_e j = \frac{\dfrac{\partial j}{\partial s}}{j} = \cos^3\phi\frac{d^2y}{dx^2} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}},$$

since

$$\cos\phi = \frac{1}{\sqrt{1+\tan^2\phi}} = \frac{1}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}.$$

Hence Theorem 5.

§ 5. *Applications of the Preceding Theorems to the Problem under Consideration.*

Let us now replace the variables x and y by τ and z respectively, and denote by U the potential function of the field of force of Theorem 5. Then

$$(41) \qquad j\cos\phi = X = -\frac{\partial U}{\partial \tau}, \quad j\sin\phi = Y = -\frac{\partial U}{\partial z},$$

$$j = \sqrt{\left(\frac{\partial U}{\partial \tau}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2},$$

and relation (37) becomes

$$(42) \qquad \frac{\dfrac{\partial j}{\partial s}}{j} = \frac{\Theta(U)}{j^3},$$

where

$$\Theta(U) = \frac{\partial^2 U}{\partial \tau \partial z}\left[\left(\frac{\partial U}{\partial z}\right)^2 - \left(\frac{\partial U}{\partial \tau}\right)^2\right] + \frac{\partial U}{\partial \tau}\cdot\frac{\partial U}{\partial z}\left[\frac{\partial^2 U}{\partial \tau^2} - \frac{\partial^2 U}{\partial z^2}\right].$$

If now $U = f(\tau, z)$ is the function (8),

$$(43) \qquad j = g = \sqrt{\left(\frac{\partial f}{\partial \tau}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2},$$

where g is the acceleration due to weight, formula (42) becomes

$$(44) \qquad \frac{\dfrac{\partial g}{\partial s}}{g} = \frac{\Theta(f)}{g^3}.$$

If, further, we denote by ξ distance measured along the north-and-south line of P_0 (positive to the south), we obtain the relation

$$(45) \qquad -\frac{\left(\dfrac{\partial g}{\partial \xi}\right)_0}{g_0} = \frac{[\Theta(f)]_0}{g_0^3},$$

in which, by Theorem 5 or relation (40), $-(\partial g/\partial\xi)_0/g_0$ is the curvature at P_0 of the line of force of the weight field which passes through P_0. Hence by Theorem 3 we have

THEOREM 8. *The curvature at P_0 of the plumb-bob locus d of P_0 is*

$$-2\frac{\left(\frac{\partial g}{\partial \xi}\right)_0}{g_0}. \tag{46}$$

Let us next put $U = \Omega$, where Ω is the function defined in (24). By (24)

$$\begin{gathered}\frac{\partial \Omega}{\partial \tau} = \frac{\partial f}{\partial \tau} - \omega^2\left(\tau - \frac{\tau_0^4}{\tau^3}\right), \quad \frac{\partial \Omega}{\partial z} = \frac{\partial f}{\partial z}, \\ \frac{\partial^2 \Omega}{\partial \tau^2} = \frac{\partial^2 f}{\partial \tau^2} - \omega^2\left(1 + 3\frac{\tau_0^4}{\tau^4}\right), \quad \frac{\partial^2 \Omega}{\partial \tau \partial z} = \frac{\partial^2 f}{\partial \tau \partial z}, \quad \frac{\partial^2 \Omega}{\partial z^2} = \frac{\partial^2 f}{\partial z^2},\end{gathered} \tag{47}$$

where f is the function (8). For the particular point P_0, whose coordinates are τ_0 and z_0, these derivatives assume the values

$$\begin{gathered}\left(\frac{\partial \Omega}{\partial \tau}\right)_0 = \left(\frac{\partial f}{\partial \tau}\right)_0, \quad \left(\frac{\partial \Omega}{\partial z}\right)_0 = \left(\frac{\partial f}{\partial z}\right)_0, \\ \left(\frac{\partial^2 \Omega}{\partial \tau^2}\right)_0 = \left(\frac{\partial^2 f}{\partial \tau^2}\right)_0 - 4\omega^2, \quad \left(\frac{\partial^2 \Omega}{\partial \tau \partial z}\right)_0 = \left(\frac{\partial^2 f}{\partial \tau \partial z}\right)_0, \\ \left(\frac{\partial^2 \Omega}{\partial z^2}\right)_0 = \left(\frac{\partial^2 f}{\partial z^2}\right)_0,\end{gathered} \tag{48}$$

whence

$$j_0 = \sqrt{\left(\frac{\partial f}{\partial \tau}\right)_0^2 + \left(\frac{\partial f}{\partial z}\right)_0^2} = g_0,$$

$$\begin{aligned}[\Theta(\Omega)]_0 &= [\Theta\,(f)]_0 - 4\omega^2\left(\frac{\partial f}{\partial \tau}\right)_0\left(\frac{\partial f}{\partial z}\right)_0 \\ &= [\Theta(f)]_0 - 4\omega^2 g_0^2 \sin\phi_0 \cos\phi_0.\end{aligned} \tag{49}$$

Therefore for the particular function Ω and the particular point P_0 we have, by (42),

$$\begin{aligned}\frac{\left(\frac{\partial j}{\partial s}\right)_0}{j_0} &= \frac{[\Theta(f)]_0}{g_0^3} - \frac{4\omega^2 \sin\phi_0 \cos\phi_0}{g_0} \\ &= -\frac{\left(\frac{\partial g}{\partial \xi}\right)_0}{g_0} - \frac{4\omega^2 \sin\phi_0 \cos\phi_0}{g_0}, \quad \text{by (45).}\end{aligned} \tag{50}$$

By Theorem 5 or relation (40) this expression (50) is the curvature at P_0 of the line of force, passing through P_0, of the positional field of force of potential function Ω. Hence by Theorems 4 and 6 we have

THEOREM 9. *The curvature at P_0 of the orthographic projection c' on the meridian plane of P_0 of the path of the falling particle is*

$$(51) \qquad -\frac{1}{3}\frac{1}{g_0}\left[\left(\frac{\partial g}{\partial \xi}\right)_0 + 4\omega^2 \sin\phi_0 \cos\phi_0\right].$$

We have now found expressions for the curvature at P_0 of the curves d and c', which pass through the point P_0, lie in the meridian plane of P_0, and have (by Theorems 1 and 2) as common tangent at P_0 the vertical P_0T of P_0 (Fig. 2). By Theorem 7 we can now find expressions for the quantities ξ_1 and ξ_2 which were defined between equations (11) and (12). For this purpose let us denote by h the height of fall P_1P_0. In the first place, let us drop perpendiculars from P_1 and C' to the line P_0T and denote by $\overline{P}_1$ and $\overline{C}'$ respectively, the feet of these perpendiculars (Fig. 5). Then the infinitesimals $P_0\overline{P}_1$,

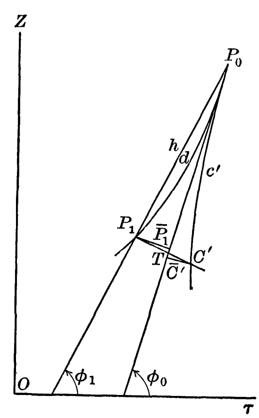

FIG. 5.

$P_0\overline{C}'$, $\overline{P}_1P_1$, $\overline{C}'C'$ differ from the infinitesimals $h = P_0P_1$, h, $\xi_2 = TP_1$, $\xi_1 = TC'$ respectively by infinitesimals of higher order. Therefore by Theorems 7, 8 and 9

$$\xi_2 = \pm \frac{1}{2}\left\{ -2\frac{\left(\frac{\partial g}{\partial \xi}\right)_0}{g_0}\right\} h^2 + \epsilon,$$

$$\xi_1 = \pm \frac{1}{2}\left\{ -\frac{1}{3g_0}\left[\left(\frac{\partial g}{\partial \xi}\right)_0 + 4\omega^2 \sin\phi_0 \cos\phi_0\right]\right\} h^2 + \epsilon',$$

where ϵ and ϵ' are infinitesimals of order higher than that of h^2. In order to remove the ambiguity in sign let us observe that since the latitute ϕ lies between $-90°$ and $+90°$, it follows that $\cos\phi > 0$, and therefore, by relation (40), $(\partial j/\partial s)/j$ and d^2y/dx^2 are either both positive or both negative. Hence it follows from (45) and Theorem 8 that for the curve d, $-(\partial g/\partial \xi)_0/g_0$ and $(d^2z/d\tau^2)_0$ are either both positive or both negative; and from (50) and Theorem 9 that for the curve c', $-1/g_0\{(\partial g/\partial \xi)_0 + 2\omega^2 \sin 2\phi_0\}$ and $(d^2z/d\tau^2)_0$ are either both positive or both negative. Furthermore, we assumed ξ_1 and ξ_2 to be positive toward the equator. Consequently if for definiteness we suppose (as shown in Fig. 5) that for curve d, $(d^2z/d\tau^2)_0 > 0$ and for curve c', $(d^2z/d\tau^2)_0 < 0$, it follows that $\xi_2 = -P_1T < 0$ and $\xi_1 = TC' > 0$. Therefore in the above expressions for ξ_2 and ξ_1 the lower signs must be used and thus we have

$$\xi_2 = \frac{\left(\frac{\partial g}{\partial \xi}\right)_0}{g_0} h^2, \tag{52}$$

$$\xi_1 = \left\{ 2\omega^2 \sin 2\phi_0 + \left(\frac{\partial g}{\partial \xi}\right)_0\right\} \frac{h^2}{6g_0}, \tag{53}$$

to terms of order not higher than the second in h, whence

$$\text{S.D.} = \xi_1 - \xi_2 = \left\{ 2\omega^2 \sin 2\phi_0 - 5\left(\frac{\partial g}{\partial \xi}\right)_0\right\} \frac{h^2}{6g_0}, \tag{54}$$

which is formula (I).

NOTE ON SOLVABLE QUINTICS.

BY PROFESSOR F. N. COLE.

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THE substance of the following paper was included several years ago in my university lecture course on the theory of