then

$$F(x) = \int_0^\infty \varphi(t)t^{x-1}dt.$$

II. If $\varphi(x)$ satisfies the conditions in Theorem I, and if $F(t) = \int_0^\infty \varphi(x)x^{t-1}dx$, then reciprocally,

$$\varphi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(t)x^{-t}dt.$$

Example:

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt,$$

$$e^{-t} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(x)t^{-x}dx \text{ for } a > 0; \ -\frac{\pi}{2} < \text{arg. } t < \frac{\pi}{2}.$$

Miss Schottenfels' paper treats of a class of functions which are self-reciprocal in the above sense of reciprocity.

H. E. SLAUGHT,
Secretary of the Chicago Section.

ON PIERPONT'S DEFINITION OF INTEGRALS.

BY PROFESSOR M. FRÉCHET.

(Read before the American Mathematical Society, December 27, 1915.)

IN the second volume of his Lectures on the Theory of Functions of Real Variables, Professor J. Pierpont has given a new definition of Lebesgue integrals. This definition is interesting in as much as it realizes an effort to adapt the previous methods of presentation of Riemann integrals to the newer Lebesgue integrals.

But unfortunately the happiness of this idea is lessened in Pierpont's work by the choice of an inappropriate definition. Professor Pierpont intended to generalize the definition of Lebesgue integrals by defining upper and lower integrals of any function $f(x)$ on any linear set E_μ. Such definitions should not, of course, be arbitrary ones, and there are some primary conditions to be fulfilled, unless these definitions are to become quite artificial and uninteresting.

For instance, it is to be expected that for any $f(x)$ and any E,

$$\overline{\int_E} f \geqq \underline{\int_E} f,$$

if the left and right sides denote the upper and lower integrals of $f(x)$ over E. And when $f \equiv 1$ these integrals ought to reduce to the upper and lower measures of E. However, if $f \equiv 1$ it will be found that those integrals are respectively the lower and upper bounds of $\Sigma\ \overline{\text{meas.}}\ \delta_n$, where $\delta_1, \delta_2, \cdots, \delta_n, \cdots$ is a "separated division of A into cells."* Hence

$$\overline{\int_E} f \leqq \underline{\int_E} f;$$

and the equality cannot hold for every E. For if, for instance, E is the interval (0, 1) and if δ_1 is a non-measurable part of E and $\delta_2 = E - \delta_1$, then $\overline{\text{meas.}}\ \delta_1 + \overline{\text{meas.}}\ \delta_2 > \text{meas. } E$. But of all the separated divisions of E, take those two the first of which consists of E itself and the second of (δ_1, δ_2); then

$$\overline{\int_E} f \leqq \text{meas. } E,$$

$$\underline{\int_E} f \geqq \overline{\text{meas.}}\ \delta_1 + \overline{\text{meas.}}\ \delta_2.$$

Thus we get a case where at the same time

$$\overline{\int_E} f < \underline{\int_E} f$$

and, though $f = 1$,

$$\underline{\int_E} f > \text{meas. } E.$$

Curiously enough, Professor Pierpont did not think it useful to mention that the inequality

$$\overline{\int_E} f - \underline{\int_E} f \geqq 0$$

* δ_i, δ_k are said to be separated when they are enclosed respectively in two measurable sets whose common part has the measure zero. Here $\overline{\text{meas.}}$ means upper measure.

should hold, as should result from his relation

$$\overline{\int_E} f - \underline{\int_E} f = \lim. \Sigma \omega_n \times \overline{\text{meas.}}\, \delta_n,$$

where ω_n is the oscillation of f in δ_n. However, it results from my example above that this relation is not always true.

The mainspring of all these difficulties is the error made in theorem 376, page 369: "Let $A = (B, C)$ be a separated division of A, then $\overline{\text{meas.}}\, A = \overline{\text{meas.}}\, B + \overline{\text{meas.}}\, C$," which is a generalization of theorem 341, page 346, "If $A = B + C$ and B, C are exterior to each other, $\overline{\text{meas.}}\, A = \overline{\text{meas.}}\, B + \overline{\text{meas.}}\, C$." The assumption made in the second line of the proof of this theorem is not altogether obvious, so that the proof is not convincing. Moreover, the theorem itself does not hold in every case; for instance, it does not when A is an interval and B a non-measurable part of A.

It is further found that the demonstration of the inequalities

$$(1) \qquad m \times \overline{\text{meas.}}\, E \leqq \underline{\int_E} f; \qquad \overline{\int_E} f \leqq M \times \overline{\text{meas.}}\, E$$

is based explicitly (§ 379, page 372) on the first theorem reproduced above and on a consequence of it which reads as follows:

$$m \times \overline{\text{meas.}}\, E \leqq \underline{S}_D; \ \overline{S}_D \leqq M \times \overline{\text{meas.}}\, E.$$

Now this consequence is easily seen to be false itself, whereas the final inequalities (1), which are correct, would have been more easily proved by showing that $m \times \overline{\text{meas.}}\, E$ and $M \times \overline{\text{meas.}}\, E$ are particular values assumed by $\underline{S}_D$ and $\overline{S}_D$ when D consists of E alone.

At any rate, many difficulties should disappear if the δ_n are to be measurable. No doubt E would then itself be measurable and the definition would not have so large an extent. However the case of the non-measurable E—which is not particularly interesting—may be easily dealt with by enclosing E in any measurable set B, letting $f = 0$ in $B - E$ and putting

$$\underline{\int_E} f = \underline{\int_B} f, \qquad \overline{\int_E} f = \overline{\int_B} f,$$

these values being obviously independent of the choice of B.

Finally, I fail to see any advantage in the use of the so-called separated divisions of E. The results are exactly the same and a useless complication is avoided if E is only divided into parts exterior to each other.

Now divide a measurable set E into a countable sequence of measurable subsets δ_i exterior to each other and denote by $\overline{\int_E} f$ and $\underline{\int_E} f$ the lower and upper bounds of $\Sigma M_i \delta_i$ and $\Sigma m_i \delta_i$, where m_i, M_i are the lower and upper bounds of f on δ_i. By these definitions, the upper integral is never smaller than the lower integral. And if $f \equiv 1$, both integrals are equal to meas. E.

This new definition is very similar to that of Riemann. The real difference is *not* as Professor Pierpont asserts for his own that it makes use of an infinite instead of a finite number of parts of E. It lies essentially in the use of measurable parts of E instead of intervals. For instance when f is bounded over E, the definition is not altered if the parts δ_i of E are assumed to be in infinite (variable) number.

When $\underline{\int_E} f = \overline{\int_E} f$ the common value of both integrals is equal to the value of the corresponding Lebesgue integral.

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REPLY TO PROFESSOR FRÉCHET'S ARTICLE.

1. REPLYING to the foregoing criticism I begin by quoting. Professor Fréchet says: " But unfortunately the happiness of this idea is lessened in Pierpont's work by the choice of an inappropriate definition. Professor Pierpont intended to generalize the definition of Lebesgue integrals by defining upper and lower integrals of any function on any linear set E_μ. Such definitions should not of course be arbitrary ones, and there are some primary conditions to be fulfilled unless these definitions are to become quite artificial and uninteresting."

The implication that the reader will easily draw is that I have not fulfilled these primary conditions and that my theory is therefore quite artificial and uninteresting. Certainly flattering to the author.