

which are parallel to the u -axis, are transformed into

$$x = (y - c_1)^2, \quad z = c_1 + c_2,$$

which are seen to be tangent to the surface S_1 . It is evident from the last equations that those one-parameter families of parabolas which lie in the planes parallel to the xy -plane have envelopes, and that no others have. These envelopes are the curves (d_1) .

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AN ELEMENTARY BOUNDARY VALUE PROBLEM.

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It is intuitively obvious that if a simple continuous curve is given in the (x, y) -plane, and a continuous distribution of values along the curve, there will exist functions of x and y which are continuous in both variables together, and which take on the prescribed values along the curve. It is the purpose of the present note to give an analytic proof of this fact, by elementary means, and, in particular, without reference to potential theory.* The problem will be treated first for the case of a rectifiable curve, then for an arbitrary Jordan curve.

Let the equations

$$x = f(s), \quad y = \varphi(s), \quad (0 \leq s \leq l),$$

define a simple closed rectifiable curve C , the variable s standing for the length of arc, and l for the total length of the curve. It is assumed that the functions $f(s)$ and $\varphi(s)$ are continuous throughout their interval of definition, and that $f(0) = f(l)$, $\varphi(0) = \varphi(l)$, but that with this exception no one pair of values (x, y) is given by two distinct values of s . Let $F(s)$ be an arbitrary continuous function defined throughout the same interval, subject to the condition that $F(0) = F(l)$.

* I understand that Mr. R. E. Gleason has had occasion to deal with a similar problem in connection with a paper recently presented to the Society; see BULLETIN, vol. 22 (1916), pp. 278-279.

We shall consider the function $J(x, y) = J_1/J_0$, where

$$J_1(x, y) = \int_0^l \frac{F(s)}{\rho^2} ds, \quad J_0(x, y) = \int_0^l \frac{ds}{\rho^2},$$

and

$$\rho = \sqrt{[x - f(s)]^2 + [y - \varphi(s)]^2}.$$

The function J is obviously defined and continuous throughout the (x, y) -plane, with the exception of the points of the curve C . We shall show that if the point (x, y) approaches a point P_0 of C , with the coordinates $(f(s_0), \varphi(s_0))$, the value of $J(x, y)$ will approach $F(s_0)$ as a limit.* There is clearly no loss of generality in assuming that s_0 is distinct from 0 and l .

We have to deal with the difference

$$J(x, y) - F(s_0) = \frac{1}{J_0} \int_0^l [F(s) - F(s_0)] \frac{ds}{\rho^2}.$$

Let ϵ be any positive quantity. Let $\delta > 0$ be chosen so that $|F(s) - F(s_0)| \leq \frac{1}{2}\epsilon$ for $|s - s_0| \leq \delta$. Then

$$\left| \int_{s_0-\delta}^{s_0+\delta} [F(s) - F(s_0)] \frac{ds}{\rho^2} \right| \leq \frac{1}{2}\epsilon J_0,$$

regardless of the position of the point (x, y) , provided only that it does not lie on C . Let γ be the minimum distance from P_0 to a point of C for which $|s - s_0| \geq \delta$. This minimum will be positive, since the distance is a continuous function of s which does not reduce to zero. Denoting by ρ_0 the distance

$$\rho_0 = \sqrt{[x - f(s_0)]^2 + [y - \varphi(s_0)]^2},$$

a quantity which depends on x and y but not on s , we can be sure that if $\rho_0 \leq \frac{1}{2}\gamma$, then $\rho \geq \frac{1}{2}\gamma$ for $|s - s_0| \geq \delta$. Consequently, if M is the maximum of $|F(s)|$,

$$\left| \int_0^{s_0-\delta} [F(s) - F(s_0)] \frac{ds}{\rho^2} \right| \leq \int_0^{s_0-\delta} 2M \cdot \frac{4ds}{\gamma^2} = \frac{8M(s_0 - \delta)}{\gamma^2};$$

* It is our purpose merely to indicate a single solution of the boundary value problem; when one solution is given, it is of course possible immediately to find infinitely many others.

and similarly

$$\left| \int_{s_0+\delta}^l [F(s) - F(s_0)] \frac{ds}{\rho^2} \right| \leq \frac{8M(l - s_0 - \delta)}{\gamma^2}.$$

We see accordingly that

$$(1) \quad |J(x, y) - F(s_0)| \leq \frac{1}{2}\epsilon + \frac{8Ml}{\gamma^2 J_0},$$

provided that (x, y) is a point at a distance from P_0 not greater than $\frac{1}{2}\gamma$, and not lying on C . It remains only to show that as (x, y) approaches P_0 , the value of J_0 becomes infinite. For if this is established, the second term on the right-hand side of (1) will be less than $\frac{1}{2}\epsilon$ when (x, y) is sufficiently near to P_0 , and it will follow that

$$|J(x, y) - F(s_0)| < \epsilon.$$

Let η be an arbitrarily small positive quantity. If $\rho_0 \leq \frac{1}{2}\eta$ and $|s - s_0| \leq \frac{1}{2}\eta$, it is certain that $\rho \leq \eta$, since it has been assumed that s represents the length of arc along the curve. For all points within a distance $\frac{1}{2}\eta$ of P_0 , therefore,

$$J_0 > \int_{s_0-\frac{1}{2}\eta}^{s_0+\frac{1}{2}\eta} \frac{ds}{\rho^2} \geq \int_{s_0-\frac{1}{2}\eta}^{s_0+\frac{1}{2}\eta} \frac{ds}{\eta^2} = \frac{1}{\eta},$$

which can be made arbitrarily large by taking η sufficiently small. This completes the proof.

Now let C be an arbitrary closed Jordan curve, given by a pair of equations

$$x = f(t), \quad y = \varphi(t), \quad 0 \leq t \leq a,$$

where the functions f and φ are continuous, and do not yield any one point twice, except for $t = 0$ and $t = a$. It will be convenient to think of f and φ as defined for all real values of t , with the period a ; they will be continuous without exception. Let $F(t)$ be an arbitrary continuous function of period a .

Let $\omega_1(\eta)$ and $\omega_2(\eta)$ be the maxima of $|f(t'') - f(t')|$ and $|\varphi(t'') - \varphi(t')|$ respectively for $|t'' - t'| \leq \eta$, and let $\omega(\eta) = \omega_1(\eta) + \omega_2(\eta)$. Then $\omega(\eta)$, defined for $\eta \geq 0$, is a function which is positive or zero, and never decreases when η increases. Furthermore, $\lim_{\eta \rightarrow 0} \omega(\eta) = 0$, because of the uniform continuity of f and φ ; and, more generally, $\omega(\eta)$ is continuous for all positive values of η , since $\omega(\eta' + \eta'')$

$\leq \omega(\eta') + \omega(\eta'')$. Let $\beta = \chi(\eta) = \omega(\eta) + \eta$. This new function is continuous, reduces to zero for $\eta = 0$, and *always increases* when η increases. To each value of $\beta \geq 0$ corresponds one and just one value of η ; the inverse function

$$\eta = \psi(\beta)$$

is itself increasing and continuous, and, in particular, $\lim_{\beta \rightarrow 0} \psi(\beta) = 0$.

We are ready now to write down a solution of the boundary value problem. We shall set $J(x, y) = J_1/J_0$, where

$$J_1(x, y) = \int_0^a \frac{F(t) dt}{[\psi(\frac{1}{2}\rho)]^2}, \quad J_0(x, y) = \int_0^a \frac{dt}{[\psi(\frac{1}{2}\rho)]^2},$$

and

$$\rho = \sqrt{[x - f(t)]^2 + [y - \varphi(t)]^2}.$$

It is true again that $J(x, y)$ is defined and continuous at all points (x, y) not lying on the given curve. Guided by the earlier demonstration, we shall begin the proof that $J(x, y)$ is a function having the desired property, by showing that J_0 becomes infinite as (x, y) approaches a point $P_0: (f(t_0), \varphi(t_0))$ of the curve.

Let β be an arbitrarily small positive quantity, and $\eta = \psi(\beta)$. If $|t - t_0| \leq \eta$, then

$$|f(t) - f(t_0)| \leq \omega_1(\eta), \quad |\varphi(t) - \varphi(t_0)| \leq \omega_2(\eta),$$

and hence the distance from P_0 to the point $(f(t), \varphi(t))$ is subject to the inequality

$$\sqrt{[f(t) - f(t_0)]^2 + [\varphi(t) - \varphi(t_0)]^2} \leq \omega(\eta) \leq \chi(\eta) = \beta.$$

Denoting by ρ_0 the distance from P_0 to the point (x, y) , we see that when

$$(2) \quad \rho_0 < \beta$$

we can be sure that $\rho < 2\beta$ for values of t in the interval just named, and hence

$$\frac{1}{2}\rho \leq \beta, \quad \psi(\frac{1}{2}\rho) \leq \psi(\beta) = \eta.$$

It follows that if (x, y) is within the neighborhood of P_0 defined by the inequality (2),

$$J_0(x, y) > \int_{t_0-\eta}^{t_0+\eta} \frac{dt}{[\psi(\frac{1}{2}\rho)]^2} \geq \int_{t_0-\eta}^{t_0+\eta} \frac{dt}{\eta^2} = \frac{2}{\eta} = \frac{2}{\psi(\beta)}.$$

As $\psi(\beta)$ approaches zero with β , the assertion with regard to J_0 is justified.

Returning to the consideration of the numerator J_1 and the quotient J , let ϵ be an arbitrarily small positive quantity, and δ a positive quantity such that $|F(t) - F(t_0)| \leq \frac{1}{2}\epsilon$ for $|t - t_0| \leq \delta$; let γ be the minimum distance from P_0 to a point of C for which $|t - t_0| \geq \delta$, a distance which is surely positive,* and finally let M be the maximum value of $|F(t)|$. If $\rho \leq \frac{1}{2}\gamma$, it is certain that $\rho \geq \frac{1}{2}\gamma$ for $|t - t_0| \geq \delta$. We see that the following inequalities hold:

$$\begin{aligned} \left| \int_{t_0-\delta}^{t_0+\delta} \frac{F(t) - F(t_0)}{[\psi(\frac{1}{2}\rho)]^2} dt \right| &\leq \frac{\epsilon}{2} \int_{t_0-\delta}^{t_0+\delta} \frac{dt}{[\psi(\frac{1}{2}\rho)]^2} < \frac{1}{2}\epsilon J_0; \\ \left| \int_0^{t_0-\delta} \frac{F(t) - F(t_0)}{[\psi(\frac{1}{2}\rho)]^2} dt \right| &\leq \int_0^{t_0-\delta} \frac{2M dt}{[\psi(\frac{1}{4}\gamma)]^2} = \frac{2M}{[\psi(\frac{1}{4}\gamma)]^2} (t_0 - \delta); \\ \left| \int_{t_0+\delta}^a \frac{F(t) - F(t_0)}{[\psi(\frac{1}{2}\rho)]^2} dt \right| &\leq \frac{2M}{[\psi(\frac{1}{4}\gamma)]^2} (a - t_0 - \delta); \end{aligned}$$

and by combination of these,

$$|J(x, y) - F(t_0)| = \left| \frac{1}{J_0} \int_0^a \frac{F(t) - F(t_0)}{[\psi(\frac{1}{2}\rho)]^2} dt \right| \leq \frac{\epsilon}{2} + \frac{2Ma}{[\psi(\frac{1}{4}\gamma)]^2 J_0}.$$

If (x, y) is sufficiently near to P_0 , the value of J_0 will be so large that the second term on the right is less than $\frac{1}{2}\epsilon$, and we shall have the inequality which establishes the theorem to be proved,

$$|J(x, y) - F(t_0)| < \epsilon.$$

It may be remarked that similar reasoning can be applied to Jordan curves that are not closed, or to a system of any finite number of Jordan curves, no two of which have a point in common.

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* We are assuming here that t is restricted to the interval $0 \leq t \leq a$, and are making for convenience the further assumption, of no essential significance, that t_0 is an interior point of the same interval.