lent special impetus to the study and development of sacred hermeneutics. The growth of this peculiar by-product of scholasticism is outlined, with references to the work on number symbolism of the Venerable Bede, St. Augustine, and numerous other prominent ecclesiastics. In particular, a remarkable example is given of a ninth century rendering of the account given in Genesis 18 of the conflict of the 318 servants of Abraham against the four kings. The classic example of the number of the beast is also studied, and it is shown that its interpretation serves to establish the date of the writing of Revelation, long in dispute, as well as affords a key to the system of number symbolism used by St. John.

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SOME REMARKABLE DETERMINANTS OF INTEGERS.

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1. THE determinants in this note are arithmetical rather than algebraic in character; their properties, not obvious by the usual reductions, follow immediately from simple considerations in the theory of numbers. Throughout, letters other than x and functional signs denote positive integers, and $[x]$ is the greatest integer in x.

2. Let $D_n\,|F(k),\ G(k)\,|$ denote the determinant of the nth order whose first and last columns are respectively $F(1)$, $F(2), \cdots, F(n)$, and $G(1), G(2), \cdots, G(n)$; and whose $(1+k)$th column $(k = 1, 2, \cdots, n-2)$ is derived from the first by prefixing k zeros and repeating in succession each element of the first $(1+k)$ times, until in all a column of n elements has been written down

$$(1) \qquad D_n\,|\,F(k),\ G(k)\,| \equiv \begin{vmatrix} F(1) & 0 & 0 & \cdots & G(1) \\ F(2) & F(1) & 0 & \cdots & G(2) \\ F(3) & F(1) & F(1) & \cdots & G(3) \\ F(4) & F(2) & F(1) & \cdots & G(4) \\ F(5) & F(2) & F(1) & \cdots & G(5) \\ F(6) & F(3) & F(2) & \cdots & G(6) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \end{vmatrix}.$$

For a reason appearing presently, $D_n|G(k),\ F(k)|$ is called the inverse of (1). A genesis of (1) is as follows: Let

$$(2)\qquad G(n) = \sum_{a=1}^{n} g(a)F([n/a]);\quad F(1) = 1;\quad F(0) = 0.$$

Then, putting in (2) successively $n = 1, 2, \cdots, n$, and solving the resulting system for $g(n)$, we find $g(n) = D_n|F(k),\ G(k)|$.

3. The more interesting determinants (1) relate to $F \equiv f'$, $G \equiv \overline{gf}'$, where the notation is

$$(3)\quad f'(n) \equiv \sum_{a=1}^{n} f(a);\ \overline{gf}(n) \equiv \sum_{n} g(d)f(n/d);\ \overline{gf}'(n) \equiv \sum_{a=1}^{n} \overline{gf}(a),$$

the $\sum_n$ extending to all divisors $1, d, \cdots, n$, of n. Clearly, $\overline{gf}(n) = \overline{fg}(n)$, and $\overline{gf}'(n)$ denotes $\Sigma g(d)f(\delta)$, the Σ referring to all d, δ such that $d\delta \leqq n$; whence, noting the number of times that d takes the particular value a, obviously

$$(4)\qquad \overline{gf}'(n) = \sum_{a=1}^{n} g(a)f'([n/a]) = \sum_{a=1}^{n} f(a)g'([n/a]).$$

Comparing (4), (2), we have, for $f(1) = g(1) = 1$, and therefore $f'(1) = g'(1) = 1$

$$(5)\quad g(n) = D_n|f'(k),\ \overline{gf}'(k)|;\ f(n) = D_n|g'(k),\ \overline{gf}'(k)|.$$

4. To limit the paper, henceforth g, f denote ψ-functions, viz., solutions of $\psi(1) = 1$, $\psi(mn) = \psi(m)\psi(n)$ for m, n relatively prime. It may be shown without difficulty that when g is given, there always exists a unique ψ-function, g_1, such that $\overline{gg_1}(1) = 1$, and $\overline{gg_1}(n) = 0$ for $n > 1$. We shall call such g, g_1 reciprocals of each other; and it is easily verified that for g, g_1 and f, f_1 reciprocals, (4) implies the inversions

$$(6)\quad f'(n) = \sum_{a=1}^{n} g_1(a)\overline{gf}'([n/a]);\ g'(n) = \sum_{a=1}^{n} f_1(a)\overline{gf}'([n/a]);$$

whence in all, as consequences of (4), the pairs of inverses

$$(7)\quad \left.\begin{aligned} f(n) &= D_n|g'(k),\ \overline{gf}'(k)| \\ f_1(n) &= D_n|\overline{gf}'(k),\ g'(k)| \end{aligned}\right\};\quad \left.\begin{aligned} g(n) &= D_n|f'(k),\ \overline{gf}'(k)| \\ g_1(n) &= D_n|\overline{gf}'(k), f'(k)| \end{aligned}\right\}.$$

Again, from (7) and the definitions directly, if $\overline{gf}(n) = h(n)$

(8) $$h(n) = \sum_n \{D_d \,|\, g'(k), \overline{gf}'(k) \,|\cdot D_{n/d} \,|\, f'(k), \overline{gf}'(k) \,|\};$$

and, noting that f, f_1 and g, g_1 are reciprocals, we have for $n > 1$

(9) $$0 = \sum_n \{D_d \,|\, g'(k), \overline{gf}'(k) \,|\cdot D_{n/d} \,|\, \overline{gf}'(k), g'(k) \,|\},$$

and the like with g, f interchanged. Also, for m, n relatively prime (but not otherwise)

(10) $$D_{mn} \,|\, f'(k), \overline{gf}'(k) \,| = D_m \,|\, f'(k), \overline{gf}'(k) \,|\cdot D_n \,|\, f'(k), \overline{gf}'(k) \,|,$$

and similarly with g, f interchanged. Hence, if $n = p^a \cdots r^b$ is the resolution of n into prime factors, $D_n \,|\, f'(k), \overline{gf}'(k) \,|$ = the product of similar determinants of respective orders $p^a, \cdots, r^b$.

5. At least one pair of the (7) will reduce to determinants of pure integers, viz., free from all functional signs f', g', $\overline{gf}'$, for such f, g as give one of $f'(k)$, $g'(k)$, and $\overline{gf}'(k)$ explicit functions of k. In the theory of numbers there are a great many such f, g: to illustrate § 4 we select in § 6 a few of the simplest. Again, for f, g given ψ-functions, it is known, or may be proved easily from first principles, that there always exists a unique ψ-function h, in general simpler than $\overline{fg}$, such that $\overline{fg}(n) = h(n)$. For specific f, g this reduction is readily effected when $n = p^a$, p prime, and hence immediately from the definition of ψ-functions for general n; similarly for the reciprocal f_1 of specific f. All the like reductions quoted in § 6 are either well known or may be thus verified with ease. We shall require the definition: a number which is divisible by no square > 1 is *simple*.

6. In illustration, consider the determinants dependent upon the following set of intimately related ψ-functions and their simpler properties: (i) $u_r(n) \equiv n^r$, whose reciprocal for $r > 0$, is $u_r(n)\mu(n)$. Also, u_0, μ are reciprocals; hence $\overline{u_0\mu}'(k) = 1$. Clearly, $u_0'(k) = k$; $u_1'(k) = \frac{1}{2}k(k+1)$. (ii) $\varpi(n) \equiv (-1)^{\lambda(n)}$, $\lambda(n)$ the *total* number of prime divisors of n. Writing $\mu(n) \times \mu(n) \equiv \mu^2(n)$; ϖ, μ^2 are reciprocals, whence

$$\overline{\varpi\mu^2}'(k) = 1.$$

Also $\overline{\mu^2 t_2} = u_0$, $\overline{u_0 \varpi} = t_2$; whence $\overline{u_0\varpi}'(k) = t_2'(k) = [\sqrt{k}]$; $\overline{\mu^2 t_2}'(k) = u_0'(k) = k$. (iii) $\mu(n) = 0$ or $\varpi(n)$ according as n is not, or is, simple (Möbius' function). (iv) $t_r(n) = 1$ or 0 according as n is, or is not, an rth power > 1. The reciprocal is $t_r(n)\mu(\sqrt[r]{n})$; and $t_r'(k) = [\sqrt[r]{k}]$.

We apply the $f(n)$, $f_1(n)$ pair in (7) to these. For $f \equiv \mu$, $g \equiv u_0$, by (i)

(11) $$\mu(n) = D_n\,|\,k,\ 1\,|; \quad 1 = D_n\,|\,1,\ k\,|.$$

For $f \equiv \mu^2$, $g \equiv t_2$, (ii) gives

(12) $$\mu^2(n) = D_n\,|\,[\sqrt{k}],\ k\,|; \quad \varpi(n) = D_n\,|\,k,\ [\sqrt{k}]\,|.$$

From the reciprocals (by (i)), $f(n) \equiv n\mu(n)$, $g(n) \equiv u_1(n)$,

(13) $$n\mu(n) = D_n\,|\,\tfrac{1}{2}k(k+1),\ 1\,|; \quad n = D_n\,|\,1,\ \tfrac{1}{2}k(k+1)\,|.$$

To complete this set we add $\varphi_r(n)$ and Gegenbauer's $\mu_r(n)$; (v) $\mu_r(n) = 0$ if n is divisible by an rth power $(r > 1)$, otherwise $= 1$. The reciprocal, γ_r, is defined by $\overline{\mu t_r} = \gamma_r$. Also, $\overline{\mu_r t_r} = u_0$; $\mu_2 = \mu^2$. (vi) $\varphi_r(n) =$ the number of integers $\leqq n^r$ that are divisible by the rth power of no prime divisor of n^r; $\varphi_1 \equiv \varphi$, the ordinary totient. The reciprocal, χ_r, is defined by $\chi_r(p^a) = (1 - p^r)$; also $\overline{u_0\varphi_r} = u_r$.

Whence, similarly to (11)–(13), we have for $f \equiv u_r$, $g \equiv t_r$; and for $f \equiv \varphi_r$, $g \equiv u_0$,

(14) $$\mu_r(n) = D_n\,|\,[\sqrt[r]{k}],\ k\,|; \quad \gamma_r(n) = D_n\,|\,k,\ [\sqrt[r]{k}]\,|.$$

(15) $$\begin{aligned} \varphi_r(n) &= D_n\,|\,k,\ (1^r + 2^r + \cdots + k^r)\,|; \\ \chi_r(n) &= D_n\,|\,(1^r + 2^r + \cdots + k^r),\ k\,|. \end{aligned}$$

For $r = 2$, (14) becomes (12); and for $r = 1$, writing $\chi_1 \equiv \chi$, (15) gives

(16) $$\varphi(n) = D_n\,|\,k,\ \tfrac{1}{2}k(k+1)\,|; \quad \chi(n) = D_n\,|\,\tfrac{1}{2}k(k+1),\ k\,|.$$

The theory of numbers furnishes a ready means of writing down any desired quantity of such results. Enough have been given for the purpose of showing how, combined with the definitions of specific ψ's, we may, as in § 7, deduce properties of the determinants themselves.

7. One class of relations between the D_n is obtained on

substituting in (8), (9), (10). E. g., by (10), (11) for m, n relatively prime: $D_{mn}|k, 1| = D_m|k, 1|\cdot D_n|k, 1|$; etc. As examples of another, we have obviously $\mu^{2s+1}(n) = \mu(n)$, $\mu^{2r}(n) = \mu^2(n)$; whence from (11), (12)

$$(17)\qquad \begin{aligned} &D_n^r|[\sqrt{k}], k| = D_n|[\sqrt{k}], k| = D_n^2|k, 1|;\\ &D_n^{2s+1}|k, 1| = D_n|k, 1|. \end{aligned}$$

Again, from (11), (13)

$$(18)\qquad D_n|1, \tfrac{1}{2}k(k+1)|\cdot D_n|k, 1| = D_n|\tfrac{1}{2}k(k+1), 1|.$$

These suffice for this kind. Curious results are found when n is prime; we illustrate, however, another kind of relation which gives less obvious properties of the D_n. Let the order n be a simple number s (defined, § 5). Then, it follows readily from the definitions of the functions that $\gamma_r(s) = \varpi(s) = \mu(s)$; $\mu_r(s) = 1$, and $\varphi_r(s) = \varpi(s)\chi_r(s)$. Hence, from (14), (12), (11)

$$(19)\qquad D_s|k, [\sqrt[r]{k}]| = D_s|k, [\sqrt{k}]| = D_s|k, 1|,$$

$$(20)\qquad D_s|[\sqrt[r]{k}], k| = 1 = D_s|1, k|;$$

and from (15), (12)

$$(21)\qquad \begin{aligned} &D_s|k, (1^r + 2^r + \cdots + k^r)|\\ &\qquad = D_s|k, [\sqrt{k}]|\cdot D_s|(1^r + 2^r + \cdots + k^r), k|, \end{aligned}$$

which, combined with (19), (20), gives further s-identities. A totally distinct kind of result may be found by using the g, g_1 pair of (7) instead of, as above, the f, f_1 pair; the D_n in this case contain ψ-functions of k. Last, isomorphic to the ψ-functions there is a class of Ψ-functions having the same formal properties as the ψ's, but based upon the resolution of n into distinct (viz., relatively prime) simple numbers, $P_1, \cdots, P_r$ (thus, $n = P_1^a \cdots P_r^b$), instead of into distinct primes; and for these Ψ's there are theorems on determinants corresponding to those for the ψ's, including the kind (19) to (21), the correspondents in the latter case being for $S = P_1 \cdots P_r$.

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