THE TWENTY-FIFTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-fifth annual meeting of the Society was held in Chicago on Friday and Saturday, December 27–28, 1918. The attendance included the following sixty members:

Professor R. C. Archibald, Professor C. S. Atchison, Professor R. P. Baker, Captain A. A. Bennett, Professor H. F. Blichfeldt, Dr. H. Blumberg, Professor J. W. Bradshaw, Professor H. T. Burgess, Professor W. D. Cairns, Professor J. A. Caparo, Professor R. D. Carmichael, Professor H. E. Cobb, Professor A. B. Coble, Professor C. E. Comstock, Mr. H. W. Curjel, Professor D. R. Curtiss, Professor L. E. Dickson, Professor E. L. Dodd, Mr. E. B. Escott, Professor W. B. Ford, Professor Tomlinson Fort, Professor O. E. Glenn, Miss A. B. Gould, Dr. J. O. Hassler, Professor Olive C. Hazlett, Professor T. H. Hildebrandt, Professor L. C. Karpinski, Professor A. M. Kenyon, Professor K. Laves, Professor S. Lefschetz, Professor A. C. Lunn, Professor Gertrude I. McCain, Professor M. McNeill, Professor W. D. MacMillan, Professor Bessie I. Miller, Professor G. A. Miller, Professor C. N. Moore, Professor E. H. Moore, Mr. E. E. Moots, Professor E. J. Moulton, Professor S. E. Rasor, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Miss Ida M. Schottenfels, Dr. A. R. Schweitzer, Professor J. B. Shaw, Professor E. B. Skinner, Professor H. E. Slaught, Professor G. W. Smith, Professor E. B. Stouffer, Professor E. J. Townsend, Professor J. N. Van der Vries, Professor E. B. Van Vleck, Professor L. G. Weld, Professor E. J. Wilczynski, Dr. C. E. Wilder, Professor F. B. Wiley, Professor R. E. Wilson, Professor C. H. Yeaton.

At the first session, held Friday afternoon, with Professor Curtiss in the chair, President L. E. Dickson presented his retiring address on "Mathematics in War Perspective." Following this address there was a joint session of the Society with the Mathematical Association of America at which President Dickson presided. The programme at this session was devoted to mathematical problems in connection with the war. The following papers were presented:

(1) Captain A. A. BENNETT: "Some mathematical features of ballistics."

(2) Professor KURT LAVES: "How the map problem was met in the war."

(3) Miss ALICE B. GOULD: "Notes concerning recent books on navigation."

(4) Professor H. L. RIETZ: "Statistics methods for preparation for war department service."

(5) Major W. D. MACMILLAN: "Ordnance problems."

(6) Lieutenant P. L. ALGER: "Practical exterior ballistics."

(7) Professor W. H. ROEVER: "The effect of the earth's rotation and curvature on the path of a projectile."

(8) Professor H. F. BLICHFELDT: "On low velocity high angle fire."

Brief accounts of these papers will be published in the report of the meeting by the Secretary of the Association in the *American Mathematical Monthly.*

On Friday evening the two organizations held a joint dinner at the Quadrangle Club, attended by sixty-eight members and friends.

At the session of the Society on Saturday, President Dickson presided in the morning and Professor Curtiss in the afternoon. The Council, which met Saturday morning, made the following announcements: There was one election to membership in the Society, Mr. V. S. Mallory, graduate student at Columbia University; and eight new applications for membership were received. The Treasurer's report was accepted, having been examined by the auditing committee. It shows a balance of $9,965.28, including the life membership fund of $6,843.46. On recommendation of the committee appointed to arrange for a summer meeting and colloquium at Chicago in 1919, it was voted to postpone that meeting for one year. The usual summer meeting, without colloquium, will be held in 1919 at a place to be determined.

At the annual election which closed Saturday noon, there were one hundred and fifty-nine votes cast. The following officers and other members of the Council were chosen:

President,	Professor FRANK MORLEY.
Vice-Presidents,	Professor G. D. BIRKHOFF, Professor FLORIAN CAJORI.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.

Librarian, Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to serve until December, 1921,

Professor H. E. HAWKES, Professor A. C. LUNN,
Professor W. A. HURWITZ, Professor C. N. MOORE.

The following papers were presented at the Saturday sessions:

(1) Professor W. B. FORD, "The sum of any series expressed as a definite integral with application to analytic continuation."

(2) Professor HARRIS HANCOCK: "On the foundations of the elliptic functions."

(3) Professor E. L. DODD: "A comparison of the median and the arithmetic mean of measurements for various laws of error."

(4) Professor DANIEL BUCHANAN: "Asymptotic orbits near the straight line equilibrium points in the problem of three finite bodies."

(5) Professor G. A. MILLER: "Groups containing a small number of sets of conjugate operators."

(6) Professor OLIVE C. HAZLETT: "A theorem on modular covariants."

(7) Mr. WAYNE SENSENIG: "Concerning the covariant theory of involutions of conics."

(8) Professor C. H. SISAM: "On surfaces containing two pencils of cubic curves."

(9) Professor E. B. STOUFFER: "On singular ruled surfaces in S_5."

(10) Dr. HENRY BLUMBERG: "A property of linear point sets."

(11) Professor E. J. WILCZYNSKI: "An application of line geometry to the theory of functions" (preliminary communication).

(12) Professor C. N. MOORE: "On the integro-derivative and some of its applications."

(13) Dr. I. A. BARNETT: "Differential equations with a continuous infinitude of variables."

(14) Dr. BARNETT: "Linear integro-differential equations with constant kernels."

(15) Dr. A. R. SCHWEITZER: "On iterative functional equations of the distributive type."

(16) Professor R. P. BAKER: "Valuation of railroad ties."

(17) Professor BAKER: "Asymptotic forms in probability."

(18) Professor HARRIS HANCOCK: "Rational and integral expressions for the roots of the biquadratic."

(19) Professor ARNOLD EMCH: "On a certain class of rational ruled surfaces."

(20) Dr. W. G. SIMON: "Two formulas of interpolation in two variables."

(21) Professor J. B. SHAW: "On triply orthogonal congruences."

(22) Professor S. LEFSCHETZ: "Real folds of abelian varieties."

(23) Dr. T. H. GRONWALL: "On surfaces of constant curvature with an equation of the form $u(x) + v(y) + w(z) = 0$."

(24) Dr. GRONWALL: "A theorem on power series with application to conformal mapping."

(25) Dr. GRONWALL: "Equipotential minimal surfaces."

(26) Dr. GRONWALL: "On Kummer's series."

(27) Professor P. J. DANIELL: "Integrals in an infinite number of dimensions."

(28) Professor C. N. MOORE: "Generalized limits in general analysis."

(29) Dr. A. R. SCHWEITZER: "On the iterative properties of an abstract group (fourth paper)."

Mr. Sensenig was introduced by Professor Glenn, and Dr. Barnett by Professor Wilczynski. Professor Hancock's second paper was read by Professor C. N. Moore. In the absence of the authors, the papers by Professor Buchanan, Mr. Sensenig, Professor Sisam, Professor Emch, Dr. Simon, Dr. Gronwall, Professor Daniell, and the first paper by Professor Hancock were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. By a method based on the calculus of residues, Professor Ford obtains an expression, in the form of a double improper integral, for the sum of any (convergent) series. When applied in particular to a power series, this expression preserves

a meaning (in general) for values of the variable lying outside the circle of convergence and thus furnishes the analytic continuation of the function defined by the series. The paper is related to a former paper of the author published in *Liouville's Journal* in 1903.

2. The differential equations through which sn u, cn u, dn u or $\wp u$, $\sqrt{\wp u - e_1}$, $\sqrt{\wp u - e_2}$, $\sqrt{\wp u - e_3}$ are defined may be put in the form of the differences of two logarithmic expressions. When integrated it may be shown that each of these expressions is a uniformly convergent power series. Thus in a direct and simple manner, without the introduction of auxiliary functions such as the Al-functions introduced in this connection by Weierstrass, Professor Hancock shows that the above functions become quotients of power series which are in fact theta functions.

3. Professor Dodd finds that with any symmetric law of error, the reliability of the median of measurements increases approximately as the square root of the number of measurements, whereas the reliability of the arithmetic mean may remain constant, as in the case of the law

$$\frac{c}{\pi} \frac{dx}{c^2 + x^2},$$

or even decrease. With certain non-Gaussian distributions, the median has a better chance than the arithmetic mean of lying close to the true value.

The so-called probability curve is at best only a good first approximation,—Karl Pearson's curves often give a much better fit. Instead of using a Chauvenet criterion to throw away measurements not conforming to this first approximation, it would appear better to ascertain more closely the law of error for a given distribution—if the number of measurements is large enough—and to base upon this a choice of arithmetic mean, median, or other function of the measurements.

4. It was shown by Lagrange that it is possible to start three finite bodies in such a way that they will describe similar ellipses. There are two configurations of equilibrium which the three bodies will maintain, viz., they are either col-

linear or lie at the vertices of an equilateral triangle. Professor Buchanan determines orbits which are asymptotic, in the sense of Poincaré, to the particular Lagrangian orbits which are described when the three bodies are collinear and move in complanar circles.

5. The object of Professor Miller's paper is to establish a few general theorems relating to the possible number of sets of conjugate operators in a group of a given order and to determine all the possible groups having a small number of complete sets of conjugate operators. When all the operators of a finite group G appear in k sets of conjugates the order of G cannot exceed the kth term diminished by unity in the series

$$2, 3, 7, 43, 1807, \ldots$$

where each term after the first is obtained by multiplying together all the preceding terms and increasing by unity the product thus obtained. In particular, a necessary and sufficient condition that a discontinuous group is of finite order is that its operators appear in a finite number of sets of conjugates under the group.

There is one and only one simple group of composite order which has the property that all of its operators appear in exactly five complete sets of conjugates, viz., the icosohedral group. The octic group, the quaternion group and the metacyclic group of order 20 are the only non-abelian groups whose operators appear in five complete sets of conjugates and whose commutator quotient group is of order 4. If all the operators besides the identity of a non-abelian group belong to two sets of conjugates under its group of isomorphisms, the order of this non-abelian group is either of the form p^α or of the form $p^\alpha - q$, p and q being prime numbers. If a non-abelian group of order p^α has the property that all of its operators besides the identity appear in three sets of conjugates under its group of isomorphisms, then its commutator subgroup is of order $p^{\alpha/2}$.

6. Professor Hazlett's paper answers a question which Dr. Sanderson raised in her thesis,* but did not answer. In her paper the fundamental theorem on the relation between formal and modular invariants for the $GF[p^n]$ enables us to construct modular covariants of a system S of binary forms in

* *Transactions Amer. Math. Society*, vol. 14 (1913), p. 490.

x and y from modular invariants of the same system S in the variables ξ and η and an additional linear form in ξ and η with coefficients y and $-x$. This is closely analogous to the situation in the algebraic theory of invariants. In the latter theory the converse is known to be true—that is, we get *all* covariants in this manner. In the case of modular invariants, however, we do not obtain all covariants, for the universal covariant $L = xy^{p^n} - yx^{p^n}$ can not be obtained as an invariant of a linear form as it vanishes whenever x and y are in the $GF[p^n]$, as we suppose the coefficients of our forms to be. In the paper referred to above, the question was raised as to whether all the covariants of a system S can be expressed as functions of L and the modular invariants of S and an additional linear form. The present paper proves that such is the case, and then, by induction, extends this theorem to a system of forms in n sets of binary variables which are transformed cogrediently.

7. Mr. Sensenig's paper contains the development and tabulation of the simultaneous concomitant set of the pencil of conics $a_x^2 + kb_x^2$, taken with the harmonic conic $(\alpha\beta x)^2$, the twenty forms being expressed as polynomials in k with coefficients which are rational expressions in the twenty forms in the known system of a_x^2 with b_x^2. The set is given in general and also in normal form. The methods are those of generalized ternary transvection, and reduction by symbolical identities. There is derived in the paper, also, the complete system of $a_x^2 + kb_x^2$ taken with a third general conic c_x^2. These forms are reduced in terms of sixty-one concomitants of a_x^2, b_x^2, c_x^2 from the system of three conics first derived by H. F. Baker.

The paper is to appear in the *American Journal of Mathematics*.

8. In a previous paper, Professor Sisam has determined the types of surfaces generated by an algebraic system of cubic curves which do not constitute a pencil. In the present paper he classifies and discusses the surfaces which contain two pencils of cubics, so that every type of algebraic surface with two or more cubic curves through each point is now determined. Each surface is represented birationally on as simple a surface as possible, the complete linear system to which the curves

corresponding to the plane or hyperplane sections belong is determined and a number of characteristic properties of the various types of surfaces are pointed out.

9. By means of a system of three ordinary linear homogeneous differential equations of the second order, Professor Stouffer studies the properties of singular ruled surfaces in space of five dimensions, including ruled surfaces in space of four dimensions. Only surfaces developable in the ordinary sense are excluded from the theory. The geometrical properties of various curves on the surface are shown, the transversal surface is obtained, and some of the relations between a surface and its transversal surface are studied.

10. Let S be any linear point set whatsoever. Let P be a point of S, i an interval enclosing P, l_i the length of i, and $m_e(i, S)$ the exterior measure (Lebesgue) of the subset of S in i. Defining the "exterior épaisseur" of S at P as the limit (if it exists) of $m_e(i, S)/l_i$ as l_i approaches 0, Dr. Blumberg proves that, except for a set of measure 0, the exterior épaisseur of S exists and is equal to 1 at every point P of S. As a consequence, every set may be represented as the sum of two sets, the first being of measure 0 and the second of exterior épaisseur 1 at every one of its points.

11. Let $w = u + iv = f(x + iy) = f(z)$ be a function of the complex variable z. Place the w plane in a position parallel to the z plane with the u and v axes parallel to the x and y axes respectively. The lines which join a point (x, y) of the z plane to the corresponding point (u, v) of the w plane form a congruence. This geometric image of a function $w = f(z)$ has been proposed by several authors, but so far no general results seem to have been obtained by following out this point of view. Professor Wilczynski has found that the congruence obtained in this way has very remarkable properties. Its focal surface is composed of two imaginary cones whose vertices are the circular points of the z plane, and whose generators correspond to the null lines of the z plane. The harmonic conjugate of the z plane with respect to the two focal cones is a real surface with real asymptotic curves whose properties have not yet been completely determined.

12. In this paper Professor C. N. Moore introduces a new generalized limit to replace the derivative of a function at points where the derivative in the ordinary sense and the so-called generalized derivative fail to exist. This limit is designated as the first integro-derivative of the rth order at the point in question and is given by the formula

$$\lim_{u\to 0}\frac{1}{u}\int_0^u dt_r\left(\frac{1}{t_r}\int_0^{t_r} dt_{r-1}\right.$$
$$\left.\times\left(\cdots\left(\frac{1}{t_2}\int_0^{t_2}\frac{f(x+t_1)-f(x-t_1)}{2t_1}dt_1\right)\cdots\right)\right).$$

The utility of this conception is illustrated by means of a theorem which establishes the summability $(C, k > r + 1)$ of the derived Fourier's series to the value of the first integro-derivative of the rth order at points where this limit exists.

13. Dr. Barnett discusses the equation

$$\frac{\partial u(\xi, z)}{\partial z} = f(\xi, z, u(\xi')),$$

where ξ, ξ' are real variables in the range $(0, 1)$, z is a real variable on $|z - z_0| \leqq \alpha$, $u(\xi')$ has the range of continuous functions for which $\max|u - u_0| \leqq \beta$, and $f(\xi, z, u)$ is a functional yielding for each (ξ, z, u) of the above ranges a real number. In the first part of the paper hypotheses are stated which are sufficient to insure the existence of a unique solution reducing for $z = z_0$ to an arbitrary continuous function $u_0(\xi)$ and having certain desirable continuity properties.

The solution is then considered as also depending on the initial conditions and theorems concerning the continuity and differentiability are proved. It is finally shown that there exists a solution to the equation

$$\frac{\partial g(z, u)}{\partial z} + \int_0^1 f(\xi, z, u)d_\xi g'(\xi, z, u) = 0,$$

where $g(z, u)$ is the functional to be determined, $f(\xi, z, u)$ is a known functional and $g'(\xi, z, u)$ is a functional related to the differential of $g(z, u)$ defined in the paper. The integration is taken in the sense of Stieltjes.

14. The most general solution of the integro-differential equation

$$(1)\qquad \frac{\partial u(\xi, z)}{\partial z} = \int_0^1 k(\xi, \eta)u(\eta, z)dz$$

has already been given by Volterra in the form

$$u(\xi, z) = w(\xi) + \int_0^1 \sum_{\rho=0}^{\infty} \frac{z^\rho}{\rho!} k_\rho(\xi, \eta)w(\eta)d\eta,$$

where the $w(\xi)$ is an arbitrary continuous function on (0, 1) and the functions k_ρ are the iterated kernels of k. Dr. Barnett shows that when the kernel k is symmetric or skew-symmetric, the solution takes on a simple form, analogous to the exponential form of the solution of a finite system of linear differential equations with constant coefficients.

The theorem for symmetric kernels is as follows:

Through the element (z_0, u_0) where z_0 is a real number and u_0 a real continuous function of ξ on (0, 1) there exists one and but one solution of equation (1), viz.,

$$u(\xi, z) = u_0(\xi) + \sum_{\rho=1}^{\infty} c_\rho \varphi_\rho(\xi)(e^{(z-z_0)/\lambda_\rho} - 1)$$

where the φ_ρ are a complete normal orthogonal set of characteristic functions of k, λ_ρ the corresponding characteristic numbers and the c_ρ the Fourier coefficients of $u_0(\xi)$ with respect to the φ_ρ, *i. e.*, $c_\rho = \int_0^1 \varphi_\rho(\eta)u_0(\eta)d\eta$.

A corresponding theorem for a special type of non-homogeneous equation is also discussed.

15. Dr. Schweitzer defines the "quasi-distributive" equations of order $k \geqq 1$; an important instance is

$$(1)\qquad \begin{aligned} &\phi[\phi(x_1, \cdots, x_{n+1}), t_1, \cdots, t_n] \\ &\qquad = \phi[x_1, \phi(x_2, t_1, \cdots, t_n), \cdots, \phi(x_{n+1}, t_1, \cdots, t_n)] \end{aligned}$$

where $n = 1, 2, 3$, etc. For $n = 1$ this relation is the associative relation. In the domain of finite or infinite abstract groups* relation (1) is always solvable. Analytically, the

* Cf. abstract, BULLETIN, Nov., 1918, pp. 58, 59. In this abstract for $k = 3$ read $k = 2$; for $k = 2$ read $k = 1$; for interpolation read interpretation.

equation (1) has the solution

$$\phi(x_1, x_2, \cdots, x_{n+1}) = \psi^{-1}[\psi(x_1) + \psi(x_2) - \Theta\{\psi(x_2) - \psi(x_3), \cdots, \psi(x_2) - \psi(x_{n+1})\}]$$

where Θ is an "arbitrary" function of $n - 1$ variables and ψ is an "arbitrary" function with an inverse. For $x_1 =$ constant in (1) this solution suggests immediately a particular solution of a certain properly distributive equation on two functions.

The equation (1) in connection with the equations

$$(2) \qquad \begin{aligned} f[\phi(x_1, t_1, t_2, \cdots, t_n), t_1, \cdots, t_n] &= x_1, \\ \phi[f(x_1, t_1, t_2, \cdots, t_n), t_1, \cdots, t_n] &= x_1, \end{aligned}$$

yields, as an eliminant in the function f, an "identity" quasi-transitive functional equation (and conversely). This result is generalized for a finite group of quasi-transitive functional equations and thus leads to the concept of a finite group of quasi-distributive functional equations and, subsequen ly, to a finite group of properly distributive equations on two functions.

16. Professor Baker gives the formula

$$\Sigma_s[\{\Sigma_v \cdot \text{Coeff. } x^s m(\Sigma_r p_r x^{l_r})^v\} \cdot \Sigma_r p_r(l_r - X + s)],$$

divided by a denominator obtained from this by replacing the last Σ_r by 1, for the most probable unexpired life of a railroad tie. Here p_r is the probability of a tie having a life l_r; Σ_r runs over all possible lives, Σ_v from $v = [X/M]$ to $v = [X/m]$ where X is the age of the road and M, m are the maximum and minimum life of a tie. Σ_s runs from $s = X - M$ to X including the first term if the result is desired immediately before the annual relaying and the last term if the result refers to a time just after this. The asymptotic form is obtained by omitting the Σ_v and replacing it by 1. This holds for all positive distributions (p_r). A large enough X to use this form has not been reached; in actual cases the result oscillates about the asymptotic amount, which is greater than the text rule of "half the average life" for a uniform distribution and less than this for a skew binomial.

Some results are as follows: taking the mean average of the two Σ_s's and $m = 5$, $M = 15$, $X = 54$ to 60 inclusive, and for the two cases of uniform distribution and skew binomial $(2 + x)^{10}$, we have

	M. A.	Asym.	Text	Oscillation
U. D.	5.266	5.25	5	1.8%
S. B.	4.037	4.032	4.167	2.2%

An oscillation of 1% is approximately $10,000,000 for the roads of the United States.

17. Professor Baker considers the integral equations

$$kf(bx) = \int_{-\infty}^{\infty} f(u)f(x-u)du \tag{1}$$

and

$$kf(bx) = \int_{0}^{x} f(u)f(x-u)du. \tag{2}$$

In volume 76 of the *Mathematische Annalen* Runge and Pólya have discussed the solution of

$$f(x) = \int_{-\infty}^{\infty} \phi(u)\phi(x-u)du.$$

Runge's method adds nothing to the discussion, but Pólya's makes the general solution depend on the functional equation

$$[\phi(s)]^2 = c\phi(as).$$

The resulting asymptotic forms include all of De Forest's based on polynomials, and also others, the simplest being $1/(1+x^2)$. This has an asymptotic neighborhood among continuous functions, and among double ended series, but not among polynomials. The equation (2) has no proper solution.

18. By making use of the associated realms of rationality, Professor Hancock is able to show that any root of a biquadratic may be expressed rationally and also integrally through any other root with coefficients that are rational and integral functions of the roots of the reducing cubic.

For example if $\sigma = \sqrt{p} + \sqrt{q} + \sqrt{r}$ and $\tau = \sqrt{p} - \sqrt{q} - \sqrt{r}$

the rather remarkable relation is

$$\tau = A_0\sigma^3 + A_1\sigma^2 + A_2\sigma + A_3,$$

where the A's are integral functions of the second degree in p.

19. It is well known that algebraic ruled surfaces may be generated by the lines joining corresponding points of an (α, β)-correspondence between the points of two algebraic curves in space. Professor Emch considers a large class of ruled surfaces defined by certain cinematic properties, which are equivalent to such a correspondence between the points of a straight line and a circle.

The parametric equations of these surfaces in cartesian coordinates are

$$x = \rho \cos \theta, \quad y = \rho \sin \theta, \quad z = (\rho - a) \cot p\theta/q,$$

where ρ and θ are the parameters, and p and q two integers which are relatively prime. It is shown that these surfaces are rational so that every plane section is a rational curve. They may be divided into two sub-classes of bifacial and unifacial surfaces according as q is even or odd.

When q is odd, the order of the surface is $2(p + q)$. In this case the directrix circle and the directrix line are respectively q-fold and $2p$-fold curves of the surface. Besides these the surface has $\frac{1}{2}(q + 1)$ real double curves of order $4p$ or $2q$, according as $2p \gtrless q$, and p real double lines.

When $q = 2s$ is even, then the order of the surface is $p + q$. The directrix circle and line are s-fold and p-fold respectively. There are moreover $\frac{1}{2}(s + 1)$ or $\frac{1}{2}s$ real double curves, according as s is odd or even; they are of order $2p$ or q, according as $2p \gtrless q$, except when s is odd; then there are $\frac{1}{2}(s - 1)$ curves of order $2p$ and one curve of order p.

The system of points of intersection of the double curves with a plane through the directrix line may be arranged according to certain cyclic groups of various orders.

These surfaces have the remarkable property that, in certain sets, they are applicable upon each other, and their intersections with a torus lead to all so-called cyclo-harmonic curves.

It is also shown how Moebius' unifacial and bifacial bands of all orders may be cut out from these surfaces.

20. In a recent paper (*Annals of Mathematics*, volume 19,

number 4) Dr. Simon gave a formula of polynomial interpolation for a continuous function of a single real variable. The formula

$$\sigma_n(x, y) = \frac{\sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i, y_j)[1 - (x_i - x)^2 - (y_j - y)^2]^n}{2\sum_{i=0}^{n} \sum_{j=0}^{n} [1 - x_i^2 - y_j^2]^n},$$

where

$$x_i = \frac{i}{\sqrt{2}n}, \quad y_j = \frac{j}{\sqrt{2}n},$$

and the function $f(x, y)$ is defined and continuous when

$$0 < a \leqq x \leqq b \leqq \frac{1}{\sqrt{2}}, \quad 0 < c \leqq y \leqq d \leqq \frac{1}{\sqrt{2}},$$

is proposed in the present paper, and is the immediate extentension of the earlier one to a continuous function of two real variables. Furthermore, it is shown that when the function which is being approximated satisfies a Lipschitz condition, the order of approximation is $1/\sqrt{n}$.

A trigonometric formula of approximation is also proposed for a continuous function of two real variables having the period 2π. This formula

$$\Sigma_n(x, y) = \frac{\sum_{i=0}^{2n-1} \sum_{j=0}^{2n-1} f(x_i, y_j)\left[\cos\frac{x_i - x}{2} \cos\frac{y_j - y}{2}\right]^{2n}}{2\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left[\cos\frac{x_i}{2} \cos\frac{y_j}{2}\right]^{2n}},$$

where the x_i and y_j are two sets of values such that $x_{i+1} - x_i = \pi/n$, $y_{j+1} - y_j = \pi/n$, is the extension of one given by Kryloff (*Bulletin des Sciences Mathématiques,* volume 41, October, 1917). Here too the order of approximation is $1/\sqrt{n}$ when the function satisfies a Lipschitz condition.

21. A congruence of curves is exemplified in the lines of a field of force, or velocity. In many questions it becomes necessary to study the rate of variation around a point of the unit vector which defines the congruence. Also in relation to the unit vector α which is tangent to the vector line (line of the congruence) at the point are other perpendicular unit vec-

tors, such as the principal normal and the binormal and others. Professor Shaw's paper is a study of the properties of such triply orthogonal sets of unit vectors. A linear vector function θ is defined which gives, for a differential displacement of the point ηds, the instantaneous axis and amount of rotation of the triply orthogonal system α, β, γ, that is $\theta(\eta)\cdot ds$. It turns out that θ is symmetric in α, β, γ and every theorem is thus true at once for each unit vector or its proper congruence. Also the curl and the divergence of each can be stated in terms of θ. The lines of equi-directed unit-tangents of the congruence are found in terms of θ. The invariant axes of θ define sets of straight lines along which the axis of rotation of the trihedral is in the direction of the displacement. Other properties of curvatures and torsions are discussed. (Cf. Rogers, *Proceedings of the Royal Irish Academy,* volume 29, section A, No. 6 (1912), pages 92–117.) Quaternion methods are used.

22. The object of Professor Lefschetz's note is to show that an abelian variety V_p of genus p and rank one can have any number of real folds represented by 2^s, $s \geqq p$. These folds form equivalent p dimensional cycles of V_p and their indices of connection are easily determined. There are Jacobi varieties of each of the p types obtained and their consideration yields a very simple proof of the theorem due to Harnack, according to which a curve of genus p can have at most $p + 1$ real branches. As a particular case of the above results, there are hyperelliptic surfaces of arbitrary divisor and rank one, with one, two or four real folds forming as many equivalent two dimensional cycles and each reducible by deformation to a ring.

23. In this paper, Dr. Gronwall shows that the determination of the surfaces indicated in the title reduces to the solution of the equation

$$f'(u)g'(v)h(w) + g'(v)h'(w)f(u) + h'(w)f'(u)g(v) = [f(u) + g(v) + h(w)]^2$$

under the condition $u + v + w = 0$. By function theoretic means, involving an extensive use of Hadamard's theory of entire functions of finite genus, it is shown that this functional equation has only one solution, which corresponds to the surfaces of revolution of constant curvature.

24. Dr. Gronwall proves the following theorem: When the power series $w(z) = \sum_0^\infty a_n z^n$ converges for $|z| < 1$, and $\sum_0^\infty n|a_n|^2$ converges, and we write $z_1 = 1 - \rho e^{\theta i}$, $z_2 = 1 - \rho e^{\theta' i}$, where $\rho > 0$ and θ and θ' vary with ρ subject only to the conditions

$$-\frac{\pi}{2} + \epsilon \leqq \theta \leqq \frac{\pi}{2} - \epsilon, \quad -\frac{\pi}{2} + \epsilon \leqq \theta' \leqq \frac{\pi}{2} - \epsilon,$$

then

$$w(z_1) - w(z_2) \to 0 \text{ as } \rho \to 0$$

uniformly in respect to θ and θ'. This theorem is useful in the study of the behavior of a conformal mapping function on the boundary of the mapped region.

25. The determination of all harmonic functions $V(x, y, z)$ such that $V(x, y, z) = \text{const.}$ defines a family of minimal surfaces is of a certain importance in hydrodynamics. In the present paper, Dr. Gronwall shows that beyond the known case $V = z - a \arctan y/x$ (and those derived from it by a change of coordinate axes), no such functions exist.

26. In this paper, Dr. Gronwall calls attention to the fact that many of the published proofs of the expansion

$$\log \Gamma(s) + \tfrac{1}{2} \log \frac{\sin \pi s}{\pi} + (s - \tfrac{1}{2})(\log 2\pi - c) = \sum_1^\infty \frac{\log n}{n\pi} \sin 2n\pi s$$

are deficient, and gives two new and elementary proofs.

27. In the *Annals of Mathematics*, June, 1918, Professor Daniell published a paper on "A general form of integral." By using the methods of that paper, he is now able to define integrals in an infinite number of dimensions, that is, not merely line integrals in an infinitely dimensional space. In a second part, the author defines a function which is of limited variation in an infinite number of dimensions and thereby is able to define a general Stieltjes integral.

28. Methods analogous to the methods for summing divergent series and integrals may be employed for defining derivatives of functions at points where the ordinary derivative fails to exist. This suggests building up a general theory which might be termed the theory of generalized limits in general analysis. In Professor C. N. Moore's paper a general theorem in this theory is established which includes as special cases the Schnee-Ford theorem with regard to the equivalence of the Cesàro and Hölder methods for summing divergent series, the corresponding theorem for integrals due to Landau, and a third theorem with regard to the equivalence of the integro-derivatives of the Cesàro and Hölder type.

29. In a former paper Dr. Schweitzer has shown that a certain "quasi-distributive" property is satisfied by the elements of an abstract group, finite or infinite. This suggests that properly distributive functional equations have solutions in the domain of abstract groups. In verification, Dr. Schweitzer shows that the relation

$$\zeta[\tau(x_1, x_2, \cdots, x_{n+1})t_1, t_2, \cdots, t_m]$$
$$= \tau[\zeta(x_1, t_1, \cdots, t_m), \cdots, \zeta(x_{n+1}, t_1, \cdots, t_m)] \quad m \geqq 1,\ n \geqq 1$$

has the solution (among others)

$$\tau(x_1, x_2, \cdots, x_{n+1}) = x_1 x_2^{-1} x_1 \cdots x_{n+1}^{-1} x_1,$$
$$\zeta(x_1, t_1, \cdots, t_m) = x_1 t_1^{p_{11}} t_2^{p_2} t_1^{p_{12}} \cdots t_m^{p_m} t_1^{p_{1m}},$$

where the p's are arbitrary integers not zero. Further, the equation on one function,

$$\tau\{x, \tau(y, z)\} = \tau\{\tau(x, y), \tau(x, z)\},$$

is satisfied by $\tau(x, y) = xy^{-1}x$. Finally, the author gives the following set of postulates (apart from closure) for an abstract group based on distributiveness:

1. $\zeta[\tau\{x, \zeta(x, y)\}, \zeta(y, z)] = \zeta(x, z)$.
2. $\zeta[\tau(x, y), z] = \tau[\zeta(x, z), \zeta(y, z)]$.
3. $\tau[x, \zeta(x, x)] = \tau[y, \zeta(y, y)]$.
4. $\tau[\zeta(x, y), x] = \zeta[\zeta(x, y), y]$.
5. $\tau[x, \tau(x, y)] = y$.

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