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IN MEMORY OF GABRIEL MARCUS GREEN.

BY PROFESSOR E. J. WILCZYNSKI.

(Read before the American Mathematical Society March 28, 1919).

GABRIEL MARCUS GREEN was born in the city of New York on October 19, 1891. He died on January 24, 1919, at Cambridge, Massachusetts. In this brief span of years he has won enduring fame. In spite of his extreme youth, we mourn in him not the promise of a genius unfulfilled, but the sad untimely loss of a great leader of proven strength whose power and insight had been fully tested, and whose actual achievements can never perish.

Green graduated from Public School No. 4 of New York in 1904 as valedictorian of his class. He then entered the high-school department of the College of the City of New York, and graduated from the College in 1911 at the head of his class. In 1909 he received the Belden Mathematical Prize, in 1909 and 1910 the Pell Medal for the highest rank in all subjects, and in 1910 and 1911 the Kenyon Prize for distinction in pure and applied mathematics. He then entered Columbia University as a graduate student, received the degree of Master of Arts in 1912, and the doctor's degree in 1913. He was a member of Phi Beta Kappa and of Sigma Xi.

While at Columbia, Green became a member of Edward Kasner's seminar. It was here that his interest in projective differential geometry was first aroused, an interest which never flagged and which dominated his whole mathematical career.

In 1913, Dr. Green acted as instructor in mathematics at the College of the City of New York. In 1914 he was appointed to an instructorship at Harvard University, and in

1916 became a member of the faculty. He was a clear, interesting, and inspiring teacher. The department was unanimous in its wish that Green should be promoted to an assistant professorship, and the matter would shortly have been laid before the corporation. But a sudden attack of influenza, followed by pneumonia, cut short this life, so valuable to science and so precious to his friends, before any action could be taken.

Green's musical gifts were hardly inferior to his mathematical talent. He was entirely self taught and was much hampered, in his earlier years, by the inadequacy of the piano which belonged to his parents. Nevertheless, diligent practice on this instrument and a dumb keyboard enabled him to acquire an extensive repertoire and a remarkable technique. His touch was delicate and his musical intuition fine. Music was a form of expression especially well adapted to his emotional and idealistic temperament.

He was short of stature and slender; his features indicated strength and refinement; his expression, always sensitive and often serious, had in it a characteristic undertone of cheerfulness and joy, the joy of a man whose faith in life had not been destroyed, and whose belief in his own powers had not been broken.

Green's first published paper is his Columbia dissertation entitled "Projective differential geometry of triple systems of surfaces," printed privately at Lancaster, Pa., in 1913. The way he wrote and published this thesis is characteristic of his independence and modesty. It had been my task for several years to keep myself informed as to the status and the progress of projective differential geometry; and I had thought that, in the United States at least, no one was likely to be engaged actively in that field without my knowledge. It was with great surprise, then, that I read of a paper with the above title presented to the American Mathematical Society at New York. The title would have aroused my interest at any time, but I had good reason to take more than ordinary interest in this paper just then. I had been engaged for some time in studying this very subject and was nearly ready to prepare my results for publication. I wrote to Green, asking him for some of the details of his work, not suspecting that it had been published already, and to my great astonishment received, a few days later, his printed thesis. This thesis made it quite unnecessary for me to publish my own work on the subject.

It is unfortunate that this thesis was printed privately; it ought to be reprinted in one of our journals, so as to render its contents generally available. The importance of the subject may be gauged by noting that it may be regarded as containing, among many other things, the theory of triply orthogonal systems, to which subject alone Darboux has devoted a volume of 567 pages.

The theory of triple systems of surfaces, as developed by Green, is based on the theory of invariants and covariants of a completely integrable system of six linear partial differential equations of the second order with one dependent and three independent variables. The subject is far too vast to be treated exhaustively in a thesis of twenty-seven pages; but the fundamental aspects of the theory are there developed and further work in this important field must inevitably set in where Green left off or else duplicate his work.

Closely connected with this thesis are two papers published in 1915 and 1916 in the *American Journal of Mathematics* on "One-parameter families of space curves and conjugate nets on a curved surface." Of course, a continuous one-parameter family of space curves generates a surface, and the properties of this surface are therefore also properties of the one-parameter family of curves. Moreover, these curves determine a second one-parameter family of curves on the surface, such that the two families together form a conjugate system. Finally each of these families determines a congruence, composed of the tangents of the curves of which it is composed. Consequently the theory of a one-parameter family of space curves, the theory of conjugate systems, the theory of surfaces, and the theory of congruences, although distinct, are all intimately connected. In formulating these connections and these distinctions analytically Green exhibits a complete mastery of the situation. The formulas developed by him in his first memoir, the paper of 1915, apply directly to any one-parameter family of curves which are not asymptotic lines on the surface S determined by them. The reason for this exception is easily understood. He uses two independent variables, u and v, one of which, namely v, is constant along one of the curves of the given one-parameter family, while the other changes along such a curve. In every case then, there presents itself a new one-parameter family of curves on the surface S, the curves $u =$ const. To simplify the analysis, Green chooses the curves $u =$ const. to be the curves on S which are conjugate to the given curves $v =$ const. This can always be done except when the given curves $v =$ const. are asymptotic curves of S, since asymptotic curves are their own conjugates. Nothing vital is lost by this procedure since the exceptional case of a one-parameter family of space curves, composed of the asymptotic lines of a surface, may be regarded as completely settled by a previously developed theory.

We have purposely discussed this paper, published in 1915, before mentioning an earlier paper of Green's on "One-parameter families of curves in the plane" (*Transactions*, 1914). There was then in existence a closely related theory whose basal features may be described in a very few words. Let $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ be the homogeneous coordinates of a point P_y in the plane. A one-parameter family of plane curves may be represented in the form

$$y^{(k)} = f^{(k)}(u, v), \qquad (k = 1, 2, 3),$$

where u is the variable which changes when P_y moves along a curve of the given one-parameter family, while v remains constant. Thus v represents the parameter of the family and changes when we pass from one curve of the family to another; the curves of the given family are the curves $v =$ const. But when formulated in this way, the fact that these functions $f^{(k)}(u, v)$ depend upon both u and v leads one to consider the curves $u =$ const. as well. The result is a study, not so much of the given one-parameter family itself, as of two one-parameter families, and the net composed of both. This was essentially the situation in which Green found the theory of one-parameter families of plane curves.

In general the second one-parameter family introduced in this way is entirely independent of the first family, and one might say that the properties of the first family are those properties of the net which remain unaltered if the second one-parameter family is changed in all possible ways. A second method of obtaining a theory pertaining entirely to the given one-parameter family of curves would be to show that there exists a second one-parameter family of plane curves completely determined by the first, and then to study the net composed of the two families. This is precisely what Green did in his theory of one-parameter families of *space* curves by associating, with the given family, the conjugate system. But strange to say, in the theory of *plane* curves, this was a far more difficult matter. The notion of conjugacy breaks down in that case; the idea which might suggest itself of associating with the given curves their orthogonal or certain other isogonal trajectories to constitute a net has to be discarded, since isogonality is a notion not admissible in projective geometry. But Green was able to conquer these difficulties by showing that every one-parameter family of plane curves determines uniquely a second one-parameter family by means of a certain purely projective relation; the theory of the canonical net, as he calls it, which is determined in this way, is then equivalent to the theory of the one-parameter family of plane curves.

In both of the papers just mentioned, Green discusses a question which is of considerable practical importance. It often happens that a theory of invariants has been developed which may be regarded as being general, in so far as it applies to every form of a certain class, but whose practical applicability nevertheless seems doubtful because the formal theory was based, not on the explicitly most general form of its class, but upon a certain canonical form to which every form of the class may be reduced. The obvious way to compute the invariants of such a form would be to reduce it to its canonical form and then to use the formulas for the invariants which the existing theory supplies. But this method cannot always be carried out because the reduction to the canonical form, while possible theoretically, may require operations, like integrations of differential equations, which cannot be carried out in practice.

Such, for instance, was the situation in the projective theory of surfaces at the time when Green wrote these papers. The general formulas were based on the assumption that the surface was referred to its asymptotic lines and, although a method had been given for computing the invariants of a surface when the asymptotic lines were not known explicitly, this method was difficult to apply to special cases. The possibility of using a different method was apparent. Although the actual determination of the asymptotic lines of a surface requires the integration of a quadratic differential equation of the first order, it is evident that the invariants of the surface must be capable of expression in terms of quantities which define the surface itself rather than its asymptotic lines; they must therefore have expressions which can be made explicit without actually integrating the differential equation of the asymptotic lines. It is Green's merit not only to have appreciated this point, which could not escape him or others, but to have shown that these expressions can be written out in a comparatively simple form. He has actually obtained these general formulas in his paper " On the theory of curved surfaces, and canonical systems in projective differential geometry," *Transactions*, 1915.

The second memoir on " Projective differential geometry of one-parameter families of space curves and conjugate nets on a curved surface " contains a number of interesting and important results. I shall speak in detail merely about one of them. It is well known that the classical metric theory of surfaces may be regarded as the invariant theory of two simultaneous quadratic differential forms which are usually written in the form

$$Edu^2 + 2Fdudv + Gdv^2$$

and

$$Ddu^2 + 2D'dudv + D''dv^2.$$

If $F = 0$, and if the ratio $E : G$ can be expressed as the product of a function of u alone by a function of v alone, the curves $u =$ const. and $v =$ const. are said to form an orthogonal isothermal system. If $D' = 0$, the curves $u =$ const. and $v =$ const. form a conjugate system. If, besides, the ratio $D : D''$ can be expressed as the product of a function of u alone by a function of v alone, the curves $u =$ const. and $v =$ const. are said to be isothermally conjugate, a terminology introduced by Bianchi and justified by the fact that these systems are related to the second differential form in the same way as isothermal orthogonal systems are to the first. Now Bianchi had shown, many years ago, that if a system of curves forms an isothermally conjugate system, this property will be preserved under all projective transformations. It is therefore a projective property. But what is this property? The definition quoted above was purely analytic, and no one had the slightest idea as to what it really meant. Nevertheless this property, whose geometric significance was entirely hidden, was entering into a continually growing chain of theorems. Could it be admitted under the circumstances that the meaning of any of these theorems was really known? Here then there was a most embarrassing situation. And since the property was known to be of a projective character, the duty of obtaining the desired interpretation clearly devolved upon projective differential geometry. In 1915 I succeeded in finding a simple algebraic relation between certain completely interpreted invariants which was satisfied if and only if the system was isothermally conjugate. But I never felt completely satisfied with my solution of the problem. A great step toward the real solution was taken by Green.

In order to make intelligible Green's condition for isothermal conjugacy, a few preliminary explanations are necessary. Let us remember in the first place that we are dealing with a conjugate system of curves on a surface S. Let P be a point of this surface; let t and t' be the tangents at P of the two curves of the conjugate systems which pass through P; let a and a' be the asymptotic tangents of the point P. Then t and t' are separated harmonically by a and a'; this follows immediately from the definition of a conjugate system. Consider the osculating plane at P of each of the two curves of the conjugate system passing through P. They intersect in a line l called the axis of the point P. Thus there is associated with every point P of S a line through P, namely, its axis. The system of ∞^2 lines associated in this way with all of the points of a surface is called the axis congruence. The lines of a congruence can be assembled into a one-parameter family of developables in two ways. Consequently there will pass through P two curves, called axis curves, such that upon each of them the axes of all of its points form developables.

The given conjugate system of curves is composed of two one-parameter families. The tangents of each of these families form a congruence of lines which has S for one sheet of its focal surface. Let P_1 be the point where the tangent t touches the second sheet of the focal surface of the congruence to which it belongs. Let us define P_{-1} similarly on t': The line P_1P_{-1} is called the ray of the point P. The system of ∞^2 rays which correspond to the ∞^2 points of S is called the ray congruence. The curves on S which correspond to the developables of the ray congruence are its ray curves. Thus we have at a general point of our surface, the two asymptotic tangents a and a', the tangents of the given conjugate system t and t', the pair of axis tangents, and the pair of ray tangents. All of these notions had been formulated by Green's predecessors, although he had been led to consider the axis congruence independently. Now Green defines his *anti-ray tangents*. They are the harmonic conjugates of the ray-curve tangents with respect to t and t'. Moreover there exists a uniquely determined pair of lines which separate both of the pairs a and a', and t and t' harmonically. These had been considered before, but Green first pointed out their great importance. He calls them the *associate conjugate* tangents:

We have now explained all of the terms in Green's theorem. *A necessary and sufficient condition that a conjugate net on a surface be isothermally conjugate is that for each point of the surface the pair of axis tangents, the pair of anti-ray tangents, and the pair of associate conjugate tangents form pairs of the same involution.* It should be noted however that there is a case, which seems to have escaped Green's attention, which is not settled by this theorem. The exceptional case arises when the axis tangents, and therefore the ray-tangents as well as the anti-ray tangents, separate the tangents of the fundamental conjugate system harmonically. It remains to find a substitute for Green's theorem to cover this case.

I have gone into such details on this point because this theorem, it seems to me, shows better than any other single one of his theorems, the penetrating power of Green's mind, and his fine feeling for mathematical elegance. Perhaps a still simpler geometric condition for isothermal conjugacy will be found; but I am inclined to believe that Green's theorem, in so far as it is applicable, is the final form for this condition.

I now proceed to another paper, also published in 1916, which shows very clearly his great power of generalization. This is his paper on "The linear dependence of functions of several variables and completely integrable systems of homogeneous linear partial differential equations."

It is a familiar fact that $n + 1$ analytic functions y_1, y_2, $\cdots$, y_n, y_{n+1} of a single variable are linearly dependent, if and only if their Wronskian is identically equal to zero. If, at the same time, the Wronskian of $y_1, y_2, \cdots, y_n$ is not zero, $y_1, y_2, \cdots, y_n$ will be fundamental solutions of a linear homogeneous differential equation

$$(D) \qquad \frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_ny = 0,$$

and the Wronskian of $y_1, \cdots, y_n$ is capable of a simple expression in terms of the coefficients of this differential equation; in fact it can be expressed in terms of p_1 alone, namely, as a simple exponential function of $\int p_1dx$, a fact first noted by Abel. A transformation of the form $y = \lambda(x)\bar{y}$ can always be made which will reduce the differential equation (D) to another one of the same form but with a vanishing coefficient for $d^{n-1}\bar{y}/dx^{n-1}$, the so-called canonical form of the differential equation, and this canonical form is unique. It then appears that the remaining coefficients of the canonical form are invariants of (D) under all transformations of the form $y = \lambda(x)\bar{y}$, where λ is an arbitrary function of x.

The analytic discussions of various problems of projective differential geometry had shown clearly in every one of the cases which had actually been investigated that a similar situation exists whenever we consider systems of linearly independent functions of several independent variables, and associate them with completely integrable systems of linear homogeneous partial differential equations. But it required deep insight and a very considerable power of generalization to reveal that the situation thus indicated by the various special theories was actually a universal one. This was accomplished by Green in a most satisfactory way without moreover restricting himself to the case of analytic functions.* It was previously known that the projective differential geometry of a k-spread in a space of any number of dimensions is equivalent to the theory of the invariants and covariants of a certain completely integrable system of linear partial differential equations. Green's theorem, however, enables us to state further that the first stage in the calculation of these invariants, namely the determination of the so-called seminvariants, may always be accomplished explicitly by the same methods which have succeeded in the special cases which have already been studied.

The latest results of Green's geometric studies are closely connected with a relation which he has called the relation R. Green's attention seems to have been called to this relation by the relation between the axis and ray congruences and by a

* In this connection paper No. 9 of the appended list is of special interest.

remarkable fact which presents itself in the theory of surfaces and which I shall recall in a few words. Let us consider a point P on a non-developable surface S and the two asymptotic curves a_1 and a_2 which cross at P. The osculating linear complexes, C_1 and C_2, of these curves have a linear congruence in common. The directrices, d_1 and d_2, of this congruence have the property that one of them, d_1, lies in the plane tangent to S at P but does not pass through P, while the other, d_2, passes through P but does not lie in the tangent plane. These two lines, d_1 and d_2, are called the directrices of the first and second kind respectively of the point P. As P moves over the surface S, they generate the corresponding directrix congruences. The curves, called directrix curves, on S which correspond to the developables of these two congruences, are the same for both; i.e., the developables of the two directrix congruences correspond to each other.

The two directrices, d_1 and d_2, of a surface point are in the relation R, but they furnish only a particular case of this relation. The general relation suggested to Green by this particular case may be explained as follows: Given a surface S and a net composed of two one-parameter families of curves on S, the curves $u =$ const. and $v =$ const.; let t_1 and t_2 be the tangents of the two curves of the net which meet at P. Given also a line l through every point P of S and not in the corresponding tangent plane. Consider the ruled surface formed by the tangents of the curves $u =$ const. along a fixed curve $v =$ const. The plane of l and t_1 will be tangent to this ruled surface at a certain point M_1 of t_1. In similar fashion we determine a point M_2 on t_2. The line M_1M_2, which we shall call l', is in the tangent plane; it is uniquely determined when the net on S and the line l through P are given. Moreover, unless the given net is conjugate, l is also uniquely determined by l'. Two lines related to one another in this way with respect to a net of curves are said by Green to be in the relation R; this nomenclature is also extended to the resulting line congruences. In the particular case when the given net is composed of the asymptotic curves of a non-developable surface, the two lines are polar reciprocals with respect to the osculating quadric of the surface point P. I shall hereafter speak of two lines l and l', which are in the relation R, as Green reciprocals of each other.

I shall mention only a few of the results which Green has obtained by using this notion. He found an elegant property of isothermal systems which may be formulated as follows: *An orthogonal net on a non-developable surface is an isothermal net, if and only if the developables of the Green reciprocal of the congruence of normals of the surface correspond to a conjugate net on the surface.* He also found the following result which emphasizes the exceptional properties of the directrix congruences. *The two directrix congruences of a surface form the only pair of congruences, whose members are Green reciprocals of each other with respect to the net of asymptotic lines of the surface, and such that the developables of the two congruences correspond to each other.*

The directrix congruences thus have undoubted claims to special attention, and a large number of questions arise when we consider the directrix congruence of the second kind as a kind of projective analogon of the congruence of normals which is so important in the metric theory of surfaces. At one point however, the analogy breaks down. The developables of the normal congruence determine on the surface a net of curves which is not only orthogonal but also conjugate, namely, the lines of curvature. But the directrix curves do not, in general, form a conjugate system. Green has succeeded in defining in purely projective fashion a new pair of congruences, Green reciprocals of each other, one of which may be regarded as being in this sense completely analogous to the normal congruence. He calls it the pseudo-normal congruence and has established its importance in most convincing fashion by a number of applications. One of these is concerned with a fundamental question which has long remained unsolved. Darboux showed, in 1880, that in the neighborhood of a regular point of a surface, the equation of the surface can be expanded in either of the two forms

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots,$$

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}\, xy(Kx^2 + Ly^2) + \cdots,$$

where x, y, and z denote non-homogeneous projective coordinates with respect to an appropriately chosen tetrahedron, and where I, J, K, and L, are absolute projective invariants of the surface. But Darboux did not explain the relation of the tetrahedron to the surface for either of these expansions. This question was settled by me about ten years ago, for the first of these two expansions. Green has now found the corresponding interpretation for the second expansion, and in this interpretation the pseudo-normal plays an important part.

Most of the results of which I have just spoken were announced by Green without proof in the form of brief notes published in the *Proceedings* of the National Academy. The proofs are given in a lengthy memoir, which has appeared in the *Transactions*, April, 1919.

My account of Green's work is far from complete. I have attempted to give only a bare outline of some of his most important work. In the six short years of his mathematical career, from 1913–1919, he enriched geometry with so many new ideas and important results as would suffice to excite our admiration if they had been spread over all of a normal life time. In his death we have suffered a heavy loss, but his life and work will continue to be, for many of us, an everlasting source of strength and inspiration.

CHICAGO,
March 29, 1919.

LIST OF GREEN'S PUBLISHED PAPERS.

1913.

1. Projective differential geometry of triple systems of surfaces. Columbia dissertation. Published privately. New Era Press, Lancaster, Pa.

1914.

2. One-parameter families of curves in the plane. *Transactions of the American Mathematical Society*, vol. 15, No. 3, pp. 277–290, July.

1915.

3. On the theory of curved surfaces and canonical systems in projective differential geometry. *Transactions of the American Mathematical Society*, vol. 16, No. 1, pp. 1–12, January.
4. Projective differential geometry of one-parameter families of space curves and conjugate nets on a curved surface. (First memoir.) *American Journal of Mathematics*, vol. 37, No. 3, pp. 215–246, July.
5. On isothermally conjugate nets of space curves. *Proceedings of the National Academy of Sciences*, vol. 1, No. 10, pp. 516–521, October.

1916.

6. On the linear dependence of functions of several variables and certain completely integrable systems of partial differential equations. *Proceedings of the National Academy of Sciences*, vol. 2, No. 4, pp. 209–214, April.
7. Projective differential geometry of one-parameter families of space curves and conjugate nets on a curved surface. (Second memoir.) *American Journal of Mathematics*, vol. 38, No. 3, pp. 287–324, July.

8. The linear dependence of functions of several variables and certain completely integrable systems of partial differential equations. *Transactions of the American Mathematical Society*, vol. 17, No. 4, pp. 483–516, October.
9. On the linear dependence of functions of one variable. *Bulletin of the American Mathematical Society*, vol. 23, No. 3, pp. 118–122, December.

1917.

10. On the general theory of curved surfaces and rectilinear congruences. *Proceedings of the National Academy of Sciences*, vol. 3, No. 10, pp. 587–592, October.
11. Some geometric characterizations of isothermal nets on a curved surface. *Transactions of the American Mathematical Society*, vol. 18, No. 4, pp. 480–488, October.

1918.

12. Note on conjugate nets with equal point invariants. *Bulletin of the American Mathematical Society*, vol. 24, No. 5, pp. 221–225, February.
13. The intersections of a straight line and hyperquadric. *Annals of Mathematics*, ser. 2, vol. 19, No. 3, pp. 207–209, March.
14. Plane nets with equal invariants. *Annals of Mathematics*, ser. 2, vol. 19, No. 4, pp. 246–250, June.
15. On certain projective generalizations of metric theorems and the curves of Darboux and Segre. *Proceedings of the National Academy of Sciences*, vol. 4, No. 11, pp. 346–349, November.

1919.

16. Memoir on the general theory of surfaces and rectilinear congruences. *Transactions of the American Mathematical Society*, vol. 20, No. 2, pp. 79–153, April.

REDUCTION OF THE ELLIPTIC ELEMENT TO THE WEIERSTRASS FORM.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society April 26, 1919.)

IN Enneper's Elliptische Functionen, page 27, may be found a method due to Weierstrass of reducing the general elliptic element to the Weierstrass form. Briefly, from

$$(1)\quad x = x_0 + \frac{\sqrt{R(x_0)}\sqrt{(4s^3 - g_2 s - g_3)} + \frac{1}{2}R'(x_0)[s - \frac{1}{24}R''(x_0)] + \frac{1}{24}R(x_0)R'''(x_0)}{2[s - \frac{1}{24}R''(x_0)]^2 - \frac{1}{2}A\cdot R(x_0)}$$