mean square deviation.) The use of this formula in the foregoing method gives the result

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^2 dx = \lim_{n=\infty} \sum_{k=-n}^{k=n} \frac{(2n)!}{(n+k)!(n-k)!} (k\Delta x)^2 \Delta x = \sigma^3 \sqrt{2\pi}.$$

Similar evaluations are obtained for

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^4 dx, \quad \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^6 dx, \text{ etc.}$$

OBERLIN, OHIO.

BÔCHER'S BOUNDARY PROBLEMS FOR DIFFERENTIAL EQUATIONS.

Leçons sur les Méthodes de Sturm dans la Théorie des Équations Différentielles Linéaires et leurs Développements Modernes, professées à la Sorbonne en 1913–1914. Par MAXIME BÔCHER. Recueillies et rédigées par GASTON JULIA. Paris, Gauthier-Villars, 1917. 6 + 118 pp.

IT can be said without fear of contradiction that what may be characterized as the *linear problem* is one of the most central in all mathematics. In algebra this problem concerns itself not only with linear forms and linear equations but also with many phases of the discussion of bilinear and quadratic forms. The results arrived at from an algebraic treatment find immediate application in geometry and mechanics. In the field of analysis the linear differential equation in one or more independent variables has always occupied a position of prime importance and in recent years the study of linear integral equations has not only forged a new and powerful tool but has also exerted a profound influence on the general trend of mathematical thought. The recent development of the theory of linear algebraic equations in an infinite number of unknowns by bridging the gap between the old algebraic field of linear equations and bilinear forms on the one hand and the analytic field of differential equations, integral equations, and bilinear forms in an infinite number of variables on the other, has given a remarkable unity to the various aspects of the general problem. In searching for the theory underlying the various forms of the linear problem, Moore has been led to the development of his General Analysis.

Perhaps the most interesting questions of relative maximum and minimum and of calculus of variations are those intimately bound up with the solutions of problems arising in these other domains. The linear differential equation as well as the linear algebraic equation owes much of its importance to its connection with applied mathematics. In fact the original discussion by Sturm was suggested by problems arising in mathematical physics, and Bôcher himself, while a student at Göttingen in the late eighties and early nineties, was led into this field by problems of a like origin which were interesting Klein and his school.

Among the methods employed during recent years in the study of boundary problems, those of asymptotic expression and successive approximations pertain more or less strictly to the field of differential equations itself. It is, however, not surprising that other disciplines have been called in to aid in the development of the theory. Among the newer tools to be used in the attack might be included integral equations, calculus of variations, passage to the limit from a finite number of approximating linear equations in a finite number of variables, and linear equations in an infinity of variables. It was, however, one of the theses stoutly defended by Bôcher that the newer methods are more artificial and less elegant than those of Sturm and that they add little to the theory.* While admitting to some extent the force of this argument, it should be pointed out that the analogies suggested by the fresh points of view have been fruitful in indicating new problems and new avenues of advance even by the old methods. And it is especially worthy of note that while the Sturmian methods proved incapable of solving boundary problems in two or more dimensions, the others lend themselves readily to such generalization.

In his article in the Encyklopädie der Mathematischen Wissenschaften, Professor Bôcher summarizes the main results attained in the discussion of boundary problems in the theory of ordinary differential equations before the present century. The paper which he read at the International Congress in 1912 outlined some of the lines of advance during the intervening years. As Harvard exchange professor at

* Cf. his lecture before the International Congress at Cambridge.

Paris, in the winter term of 1913–14, it was especially fitting that he lay stress on the notions and methods of Sturm, who in the same city more than three quarters of a century earlier had in a brilliant memoir laid the foundation of this fundamental theory. Soon after his student days Bôcher had conceived the idea of writing a treatise on boundary problems as an elaboration of his encyclopedia article and had entered into a contract with Teubner for its publication. A considerable portion of this manuscript in German was left among his papers, but the contract had been cancelled a few years before his untimely death. The present volume embodies much of the material he had prepared for the other project.

The contributions of Americans to this field added to the felicity of the choice of the topic for the Paris lectures. In addition to the many contributions of the lecturer and his students, several other Americans, who a decade or more ago came under the influence of Hilbert, have done investigation in this field. Besides the notable advances made in the several papers by Birkhoff, other names such as Mason, Westfall, Kellogg, and Hurwitz might be mentioned in this connection. It is worthy of note also that a considerable proportion of the articles* dealing with this topic and published

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* Among the papers touching on this field published since Bôcher's survey in 1912 and not referred to in his Leçons are the following: J. Tamarkine, "Sur quelques points de la théorie des équations différentielles linéaires ordinaires et sur la généralization de la séries de Fourier," *Rend. Cir. Mat. di Pal.*, vol. 34 (1912), pp. 345–382. Also see vol. 37 (1914), pp. 376–378 of same journal. G. D. Birkhoff, "Note on the expansion problems of ordinary linear differential equations," *Rend. Cir. Mat. di Pal.*, vol. 36 (1913), pp. 115–126. G. Hamel, "Ueber die lineare Differentialgleichung zweiter Ordnung mit periodischen Koeffizienten," *Math. Ann.*, vol. 73 (1913), pp. 371–412; also "Ueber das infinitäre Verhalten der Integrale einer linearen Differentialgleichung zweiter Ordnung, wenn die characteristische Gleichung zwei gleiche Wurzeln hat," *Mathematische Zeitschrift*, vol. 1 (1918), pp. 220–228. J. Yoshikawa, "Miszellen aus dem Gebiete der Oszillationsaufgaben," *Memoirs of the College of Science and Engineering, Kyoto Imperial University*, vol. 5 (1913), pp. 97–115. O. Haupt, "Ueber eine Methode zum Beweise von Oszillationstheoremen," *Math. Ann.*, vol. 76 (1914), pp. 67–104. L. Lichtenstein, "Zur Analysis der unendlichvielen Variabeln. Entwicklungssätze der Theorie gewöhnlicher linearer Differentialgleichungen zweiter Ordnung," *Rend. Cir. Mat. di Pal.*, vol. 38 (1914), pp. 113–166. D. Jackson, "Algebraic properties of self-adjoint systems," *Trans. Amer. Math. Soc.*, vol. 17 (1916), pp. 418–424. T. Fort, "Linear Difference and Differential Equations," *Amer. Jour. of Math.*, vol. 39 (1917), pp. 1–26; also "Some theorems of comparison and oscillation," *Bull. Amer. Math. Soc.*, vol. 24 (1918), pp. 330–334. R. D. Carmichael, "Comparison theorems for homogeneous linear differential equations of general order," *Annals of Math.*, vol. 19 (1918), pp. 159–171. H. J. Ettlinger, "Existence theorems for the general real self-adjoint

since these lectures were prepared, are from the pens of Americans.

The relation of this volume to the rest of Bôcher's work has already been discussed in this BULLETIN* by Birkhoff in his general survey and estimate of the author's contributions to mathematics. The particular task which he has set himself in the book under review is that of expounding the theory of the linear differential equation, keeping in mind not only the spirit of Sturm's work but, so far as possible, the letter also. Choosing to neglect some of the more direct and powerful but less elegant methods, he has given an admirable exposition of the analogy of the problems of linear differential equations to those of the purely algebraic linear equations.

Ever since Sturm was inspired by these analogies to undertake his investigations, an advance in the linear problem of algebra or of analysis has generally suggested a corresponding one in the other field. Bôcher's contributions to the literature of both the algebraic and analytic problems made it especially fitting that he undertake this presentation. It should, however, be pointed out that it is purely the parallelism between the two theories which he discusses. No use is made of either theory in the actual development of the other. There is, for example, no hint of the possibility of obtaining results by passing from the algebraic to the transcendental on letting the number of difference equations approximating to the differential equation increase without limit.

Material from the published memoirs of Bôcher plays no inconsiderable part in the subject matter of this volume. And as Birkhoff has already pointed out, there are two important

linear system of the second order," *Trans. Amer. Math. Soc.*, vol. 19 (1918), pp. 79–96. W. B. Fite, "Concerning the zeros of the solutions of certain differential equations," *Trans. Amer. Math. Soc.*, vol. 19 (1918), pp. 341–352; also "The relation between the zeros of a solution of a linear homogeneous differential equation and those of its derivative," *Annals of Math.*, vol. 18 (1917), pp. 214–220. O. D. Kellogg, "Interpolation preperties of orthogonal sets of solutions of differential equations, *Amer. Jour. of Math.*, vol. 40 (1918), pp. 225–234. R. G. D. Richardson, "Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order," *Amer. Jour. of Math.*, vol. 40 (1918), pp. 283–316. C. E. Wilder, "Problems in the theory of ordinary linear differential equations with auxiliary conditions at more than two points," *Trans. Amer. Math. Soc.*, vol. 19 (1918), pp. 157–166; also see vol. 18 (1917), pp. 415–422 of same journal.

See also abstracts of two papers read before the meeting of the Amer. Math. Society in September 1919 by C. N. Reynolds, *Bull. Amer. Math. Soc.*, vol. 26 (1919), no. 2.

* February, 1919.

theorems which appear here for the first time. Below we shall discuss these theorems, one of which is concerned with an extension of the method of successive approximation and the other with the functional dependence of the solution on the coefficients of the equation and of the boundary conditions. A careful reading of these lectures gives the distinct impression also that a further filling in of gaps and a general rounding out of the theory is one of their admirable features.

The reader cannot fail to obtain from this volume an excellent insight into one of the fundamental bases of mathematical theory. It has the virtues and the faults of the lecture form, being interesting and suggestive but neither encyclopedic nor adapted for reference. It suffers from the fact that it was compiled by another; one notes the simplicity and elegance of ideas which were characteristic of Bôcher, but misses at times his lucidity of exposition. Sometimes the exact conditions under which a theorem is being developed are not easy to locate (pages 40, 71, 110) and it is occasionally difficult to find the formulation of results (pages 41, 42, 62, 81). The arrangement of material is not always the happiest and the passage from one topic to another is sometimes more abrupt than the division into sections would indicate. It may be that the war is in large measure responsible for a standard of proof-reading and typography somewhat lower than the very excellent one hitherto attained in the Borel series.

Before attempting to outline the argument of the text it may be well to consider briefly some of the properties of solutions of the special differential equation.

$$(1) \qquad u'' + \lambda u \equiv d^2u/dx^2 + \lambda u = 0$$

and its approximating difference equations. For this special problem we shall not only, as Bôcher would do, set up the parallelism of the two theories but we shall go beyond him in exhibiting the analogy of the actual mechanism and in indicating how by passing to the limit one theory goes into the other. The results are fairly typical of the general homogeneous equation of the second order and will serve as an illuminating introduction to the general theory.

There are two linearly independent solutions of (1), viz.,

$\sin \sqrt{\lambda}x$ and $\cos \sqrt{\lambda}x$, depending on the parameter λ and giving as the most general solution $u = c_1 \sin \sqrt{\lambda}x + c_2 \cos \sqrt{\lambda}x$. The zeros of any two linearly independent solutions separate each other. If now one boundary condition is imposed, there can be only one linearly independent solution; for example, when $u(0) = 0$, the solution becomes $u = c_1 \sin \sqrt{\lambda}x$, having zeros at $\pm n\pi/\sqrt{\lambda}$. With increase of λ these zeros, or those of any solution, move closer together. It is readily seen that a second boundary condition can be imposed only in exceptional cases; the condition $u(1) = 0$ can, for example, be satisfied only if $\lambda = \pi^2, (2\pi)^2, \cdots$. The oscillation theorem for equation (1) under the boundary conditions

$$u(0) = u(1) = 0 \tag{2}$$

may be stated as follows: There exists one and only one characteristic number λ for which there is a non-identically vanishing characteristic solution of (1), (2) oscillating n times, that is, having n zeros in the interval. The infinity of positive characteristic numbers for $n = 1, 2, 3, \cdots$ diverges to infinity.

By dividing the interval $(0, 1)$ into n equal parts and denoting by u_i the value at $x = i/n$, we can set up the algebraic equations approximating to (1) or (1) and (2). In the latter case they are

$$n^2(u_{i-1} - 2u_i + u_{i+1}) = -\lambda u_i, \qquad u_0 = 0,\ u_n = 0 \quad (i = 1, \cdots, n-1). \tag{3}$$

Essentially we have here $n - 1$ homogeneous equations in $n - 1$ unknowns which will have a solution only when λ is one of the $n - 1$ roots of a determinant formed from the coefficients of the u's. The λ-matrix of the coefficients which is the type met with in studying pairs of bilinear or quadratic forms is here symmetric. We can call the roots characteristic numbers and state for the corresponding solutions a theorem of oscillation analogous to that for the differential equation.

If, in the approximation equations (3), n is given the value 5 and the u's on the left considered to be a different set of variables from the u's on the right, and the former solved in terms of the latter, these equations take the form

$$
\begin{aligned}
u_1 &= \frac{\lambda}{25}(u_0 + \tfrac{4}{5}u_1 + \tfrac{3}{5}u_2 + \tfrac{2}{5}u_3 + \tfrac{1}{5}u_4 + 0u_5),\\
u_2 &= \frac{\lambda}{25}(0u_0 + \tfrac{3}{5}u_1 + \tfrac{6}{5}u_2 + \tfrac{4}{5}u_3 + \tfrac{2}{5}u_4 + 0u_5),\\
u_3 &= \frac{\lambda}{25}(0u_0 + \tfrac{2}{5}u_1 + \tfrac{4}{5}u_2 + \tfrac{6}{5}u_3 + \tfrac{3}{5}u_4 + 0u_5),\\
u_4 &= \frac{\lambda}{25}(0u_0 + \tfrac{1}{5}u_1 + \tfrac{2}{5}u_2 + \tfrac{2}{5}u_3 + \tfrac{4}{5}u_4 + u_5),\\
u_0 &= 0, \qquad u_5 = 0.
\end{aligned} \tag{4}
$$

On denoting the range i/n $(i = 0, 1, 2, 3, 4, 5)$ by x_i and also by ξ_i, the matrix of coefficients on the right may be written

$$
\begin{aligned}
K(x_i, \xi_j) &= (1 - \xi_j)x_i, \text{ for } x_i \leqq \xi_j;\\
K(x_i, \xi_j) &= (1 - x_i)\,\xi_j, \text{ for } x_i \geqq \xi_j,
\end{aligned} \tag{5}
$$

and the solution (4) has the form

$$
u_j = \frac{\lambda}{5}\sum_{i=1}^{5} K(x_i, \xi_j)u_i \qquad (j = 0, \cdots, 5). \tag{4'}
$$

It will be noted that $K(x_i, \xi_j)$, which may be called the Green's function, is a symmetric matrix $[K(x_i, \xi_j) = K(\xi_j, x_i)]$, and further that the first difference quotient $5[K(x_{i+1}, \xi_j) - K(x_i, \xi_j)]$ for each j is one constant for $i \leqq j$ and another constant for $i > j$ and that the difference of these constants is in all cases unity. In other words, for $x_i = \xi_j$ the first difference of the matrix has a discontinuity equal to unity.

We may also set up a Green's function $K(x, \xi)$ for the differential equation (1) having the following properties: $K(x, \xi)$

(*a*) is symmetric in the variables (x, ξ),

(*b*) is continuous together with its first derivative, except that the latter has a break of 1 for $x = \xi$, and

(*c*) satisfies the relation

$$
u(x) = \lambda \int_0^1 K(x, \xi)\, u(\xi)\, d\xi. \tag{6}
$$

This integral equation (6) is equivalent to the differential system (1), (2) and bears to it the same relation as equations

(4) and (4′) do to equations (3). The Green's function

$$(7)\qquad \begin{aligned} K(x, \xi) &= (1 - \xi)x \text{ for } x \leqq \xi, \\ K(x, \xi) &= (1 - x)\xi \text{ for } x \geqq \xi \end{aligned}$$

has essentially the same form (5) as that for the approximating algebraic equations (3).

The λ-matrix of any set of approximating equations such as (3) is equivalent to that of the corresponding set such as (4), both sets being taken as homogeneous. For any value of n, each of the characteristic numbers of (3) is equal to the corresponding one of (4). When n increases, each of the former approaches one of the infinite set of characteristic numbers $(n\pi)^2$ of (1), while each of the latter approaches one of the same infinite set of characteristic numbers of (6). The λ-determinant of (4) has for limit the symmetric Fredholm λ-determinant of (6). Thus by a passage to the limit $n = \infty$ the algebraic problem goes over completely into the transcendental.

After this consideration of a special problem, we are in a position to sketch the methods and results of the Leçons. Chapter II concerns itself with the parallelism between the linear differential and the linear algebraic systems. To a square array $|a_{ij}|$ there corresponds a set of linear homogeneous equations

$$(8)\qquad a_{i1}\xi_1 + c_{i2}\,\xi_2 + \cdots + a_{in}\xi_n = 0 \quad (i = 1, 2, \cdots, n)$$

and also another closely related adjoint set

$$(9)\qquad a_{1j}\eta_1 + a_{2j}\eta_2 + \cdots + a_{nj}\eta_n = 0 \quad (j = 1, 2, \cdots, n).$$

The rank of the determinant $|a_{ij}|$ being p, there are $n - p$ linearly independent sets of solutions of each set of equations. If $p = n$, there is only the identically zero solution and the systems are each incompatible. In this case the corresponding sets of non-homogeneous equations

$$(8')\quad \sum_{j=1}^{n} a_{ij}\xi_j = \gamma_i, \qquad (9')\quad \sum_{i=1}^{n} a_{ij}\eta_i = \delta_j$$

have each a unique solution. If $n - p = m \neq 0$ and the homogeneous sets (8) and (9) both have solutions, one must impose m linear conditions on the γ's (or δ's) in order that the non-homogeneous system (8′) (or 9′) have at the same time a solution. The general solution of (8′) is the sum of a particular solution and of the most general solution of (8), the latter depending on $n - p$ constants.

Turning now to a discussion of the most general linear differential equation of the second order,

$$(10)\quad L(u) \equiv u'' + Pu' + Qu \equiv \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = R,$$

where P, Q, and R are continuous functions in an interval A, B, we observe that if no boundary conditions are imposed, the solutions can be approximated by $n - 1$ non-homogeneous difference equations which take the form

$$(11)\qquad A_iu_{i-1} + B_iu_i + C_iu_{i+1} = D_i \quad (i = 1, \cdots, n - 1),$$

in which the $n + 1$ variables u are the values of the function at equally spaced points in the interval. Since there are two more variables than equations, the values of two of them may generally be assigned at random; in other words, the solution depends linearly on two arbitrary constants. In order to arrive at unique results, it would be necessary to impose two more linear conditions. As examples of such conditions the following may be cited:

(*a*) The values of the function or its derivative at two points a, b, of the interval may be pre-assigned; or more generally two linear conditions of the form

$$(12)\quad U_i(u) \equiv \alpha_{i1}u(a) + \alpha_{i2}u'(a) + \beta_{i1}u(b) + \beta_{i2}u'(b) = \delta_i \qquad (i = 1, 2),$$

may be imposed. More generally still these conditions may concern themselves with the values of u and u' at any number of points in the interval.

(*b*) Given the functions f_{i1}, f_{i2}, the solutions may be subject to the integral conditions

$$u_i(u) = \int_A^B [f_{i1}u(x) + f_{i2}u'(x)]dx = 0.$$

(*c*) There may be a combination of the types (*a*) and (*b*).

The equations (11) with two linear conditions such as (12) correspond to one of the sets (8′), (9′).

To the homogeneous equations (8) corresponds the homogeneous differential equation

$$L(u) \equiv \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0, \tag{13}$$

together with two linear homogeneous equations of condition

$$U_i(u) \equiv \alpha_{i1}u(a) + \alpha_{i2}u'(a) + \beta_{i1}u(b) + \beta_{i2}u'(b) = 0 \quad (i = 1, 2). \tag{14}$$

The differential equation adjoint to (10) is

$$M(v) \equiv \frac{d^2v}{dx^2} - \frac{d(Pv)}{dx} + Qv = R \tag{15}$$

and to (13) is

$$M(v) \equiv \frac{d^2v}{dx^2} - \frac{d(Pv)}{dx} + Qv = 0. \tag{16}$$

Similarly we may have boundary conditions adjoint to (12) or (14). In the latter case they are

$$V_i(v) = 0 \qquad (i = 1, 2), \tag{17}$$

and we have the formula

$$vL(u) - uM(v) = \frac{d}{dx}(Puv - Pv' + u'v). \tag{18}$$

The two systems (13), (14) and (16), (17) have the same number of linearly independent solutions or are both incompatible. The approximation equations of the two will bear to one another the same relation as (8) and its adjoint set (9), and the formula (18) has its analogue in a bilinear form involving the ξ's and η's. Precisely as in the case of the algebraic difference equations, the incompatibility of (13), (14) entails a unique solution of (10), (12) and of (13), (12) and of (10), (14). In case the system (13), (14) is compatible, a necessary and sufficient condition that there be a solution of the non-homogeneous system (10), (12) also is that certain linear equations of condition be imposed on R and the δ's, while the general solution is the sum of a particular solution and of the most general solution of (13), (14).

Every differential equation of the second order may, by a change of variable, be put into a form which is self-adjoint, that is, such that (10) and (15) and also (13) and (16) coincide. The type most important in application is that where the boundary conditions are also self-adjoint; (14) and (17) would in that case be identical. This corresponds in the algebraic cases (8), (9) to a symmetric array $|a_{ij}|$.

The discussion of the simplest cases of the real solutions of the self-adjoint equation together with the behaviors of their zeros forms the topic of Chapter III. The properties of solutions of the most general homogeneous equation of this type, which can be taken in the form

$$(pu')' + gu = 0, \tag{19}$$

follow closely those concerning the solutions of (1) which were written in explicit form. The well-known theorem of Sturm concerning the alternation of the zeros of two linearly independent solutions u_1, u_2 of (19) follows from the readily established formula

$$\frac{d(u_1/u_2)}{dx} = \frac{u_2u_1' - u_1u_2'}{u_2^2} = \frac{\text{constant}}{pu_2^2}.$$

Conditions under which this separation theorem holds for two linear combinations of u and u' such as

$$\psi_1 = \varphi_1(x)u + \varphi_2(x)u', \quad \psi_2 = \varphi_3(x)u + \varphi_4(x)u'$$

are readily obtained. Of course, corresponding theorems for the approximating algebraic equations can also be deduced; but they are not of special interest.

The solutions of (19) being continuous functionals of p and g, we should expect that by proper choice of variation in these coefficients the zeros of $u(x)$ would move closer together as in the special case of equation (1), and this is a result easily established. Denoting by u_1 the solution corresponding to coefficients p_1 and g_1 in (19) and by u_2 that corresponding to p_2 and g_2, it follows from the identity, due to Picone,

$$\frac{d}{dx}\left[\frac{u_1}{u_2}(p_1u_2u_1' - p_2u_1u_2')\right]$$
$$= (g_2 - g_1)u_1^2 + (p_1 - p_2)u_1'^2 + p_2\left(u_1' - u_2'\frac{u_1}{u_2}\right)^2,$$

that, in an interval between two consecutive zeros x_1 and x_2 of u_1, a necessary condition that there exist a solution $u_2 \neq 0$ is that at least one of the expressions $p_1 - p_2$, $g_2 - g_1$ be positive. For otherwise on integrating the identity between x_1 and x_2, the two sides give opposite signs. Roughly speaking, it is true that if p decreases or g increases the zeros move closer together. An analogy can be found for the corresponding approximating difference equations.

By means of this comparison theorem the author easily finds upper and lower bounds for the lengths of intervals of oscillation by solving an equation with constant coefficients greater or less than p and g. For example, if $\gamma_1 = \min p > 0$, $\gamma_2 = \max g$, on solving

$$u'' + \frac{\gamma_2}{\gamma_1} u = 0,$$

it is found that if $\gamma_2 < 0$ the exponential result precludes the possibility of an oscillating solution of (19), while if $\gamma_2 > 0$, $\sin \sqrt{(\gamma_2/\gamma_1)}\, x$ gives a lower bound $\pi/\sqrt{\gamma_2/\gamma_1}$ for the interval between the zeros of that equation.

The most important special case which arises in applications is when $g = q + \lambda k$; that is, when (19) takes the form

$$(20) \qquad (pu')' + (q + \lambda k)u = 0.$$

If p, q and k remain fixed and the parameter λ varies, the theorem of comparison just enunciated can be applied directly in case $k \geqq 0$ or $k \leqq 0$. In case $q \leqq 0$ and k changes sign it may be applied by dividing the equation through by $|\lambda|$. The remaining case where no restrictions are put on q and k involves complex solutions and has been worked out in detail only since Bôcher's lectures were given.*

Let us now impose one condition on the solution $u(x)$ of (20); for example,

$$(21) \qquad c_1 u(a) + c_2 u'(a) = 0.$$

(This may, by a change of variable, be reduced to $u(a) = 0$ or $u'(a) = 0$.) By proper variations of p, q and λk the lengths of the intervals may be decreased and the zeros of u and u' made to move in toward $x = a$. In particular, when p, q and k are held fixed, with increase of $|\lambda|$ the number of

* Cf. Haupt and Richardson cited above.

zeros of $u(x)$ in the interval a, b can be made to increase without limit. Further there is a set of values of $\lambda = \lambda_1, \lambda_2, \cdots$ (characteristic numbers), which diverge to infinity, to which correspond the characteristic solutions $u(x)$ satisfying (21) and a second linear condition

$$c_3 u(b) + c_4 u'(b) = 0. \tag{21'}$$

In case $k \geqq 0$ the oscillation theorem for the system (20), (21), (21′) will be approximately that for the special system (1), (2); but in case k has both signs the result must be so modified as to prescribe two characteristic numbers (one positive and the other negative) corresponding to which are solutions having any given number of zeros in the interval.

Besides a discussion of conditions under which the characteristic numbers of a differential equation will be real and the derivation of an oscillation theorem for a simple case of n differential equations each with n parameters, Chapter IV contains a detailed treatment of the oscillation theorem for the interesting periodic conditions

$$u(a) = u(b), \qquad u'(a) = u'(b). \tag{22}$$

It may be remarked that when more general boundary conditions of the type (14) are imposed, it is customary to follow Hilbert in reducing them to a small number of normal forms and in discussing these in sequence. It may also be noted that the theorem of oscillation derived by Bôcher holds more generally for all boundary conditions belonging to the same normal form as (22). For $k \geqq 0$ the theorem is precisely that which one would surmise from a slight consideration of the special equation (1) under these same conditions (22), viz.: corresponding to each even number of oscillations there are two characteristic solutions, to each odd number there are none, and to zero there is one.

Chapter V begins with a consideration of the existence and fundamental properties of the Green's function for differential equations of the nth order. Since the case $n = 2$ is typical, we may here confine ourselves to the equations already discussed. When the homogeneous system (13), (14) is incompatible, a Green's function $G(x, \xi)$ having the properties (a), (b), (c) enumerated above may be set up and it furnishes as a unique solution of (10), (14) the formula

$$u(x) = \int_a^b R(x) G(x, \xi) d\xi. \tag{23}$$

The Green's function for the adjoint system (15), (17) turns out to be $G(\xi, x)$, so that for a self-adjoint system it is symmetric in x and ξ, as we saw in the special form (7) set up for the system (1), (2). When a solution of the system (10), (12) is desired, the formula (23) must be modified by the addition of certain auxiliary terms. While no mention is made of the fact in the lectures, these theorems have an analogue in the theory of algebraic equations.

The same sort of argument as establishes (23) leads to a proof of the equivalence of the non-homogeneous differential system (10), (12) with the non-homogeneous integral equation

$$u(x) = f(x) + \int_a^b R(x)G(x, \xi)u(\xi)d\xi,$$

where $f(x)$ is the sum of certain auxiliary terms. In case the differential system is homogeneous, the term $f(x)$ is zero and (13), (14) is equivalent to the corresponding homogeneous integral equation

$$u(x) = \int_a^b R(x)G(x, \xi)u(\xi)d\xi$$

of which (6) is a special case.

In Chapter I the author proves by the method of successive approximations the usual existence theorem for the solution of the non-homogeneous equation (10) under the one-point boundary condition

$$(24) \qquad u(c) = \gamma, \qquad u'(c) = \gamma'$$

by showing that the series

$$v_1 + v_2 + v_3 + \cdots; \qquad v_1' + v_2' + v_3' + \cdots$$

approximating to u and u' are uniformly convergent. This is done by establishing the inequality

$$\left.\begin{matrix} |v_n| \\ |v_n'| \end{matrix}\right\} \leqq \frac{CL^{n-2}(2M)^{n-2}(x-c)^{n-2}}{(n-2)!},$$

where C is greater than $|v_1|$, $|v_1'|$, $|v_2|$, $|v_2'|$; M greater than $|P|$, $|Q|$, $|R|$, $|\gamma|$, $|\gamma'|$; and L greater than both unity and the length of the interval.

One of the contributions of Bôcher first appearing in these lectures is the answer to the question as to the nature of the dependence of u on the coefficients P, Q, R, and the constants γ, γ'. He shows that the solution u is a continuous *functional* $F(P, Q, R, \gamma, \gamma')$, and moreover that this is true even when the three arguments P, Q, R have a finite number of discontinuities. It may be assumed that as in the case of functions the sum or product of two continuous functionals is a continuous functional as is also the integral of a functional. The reasoning of Bôcher, which is characteristically simple and elegant, then proceeds as follows. Since each v and v' is obtained by a finite number of integrations of explicit sums and products of P, Q, R, γ, γ', and of specified continuous functions of x, it is a continuous functional of the arguments. The finite sums $\sum_1^{n_1-1} v_n$, $\sum_1^{n_1-1} v_n'$ will then also be continuous functionals for any definite index n_1. But for the functional field limited by the assumed value of the constant M, and for an index N_1 large enough, $\left|\sum_{n_1}^{\infty} v_n\right|$ and $\left|\sum_{n_1}^{\infty} v_n'\right|$ will be less than any preassigned $\epsilon > 0$ for $n_1 > N_1$. This establishes the theorem.

Returning to the topic of successive approximations at the end of Chapter V, Bôcher by an application of the Green's function fulfils the promise made in his International Congress lecture of considering a generalization of the theory of successive approximations. Taking the differential system in the form

$$(25) \qquad L'(u) = L''(u) + R, \qquad U_i'(u) = U_i''(u) + \delta_i,$$

where L' and L'' are homogeneous linear differential expressions of orders n and $m < n$ respectively and $U_i'(u)$ and $U_i''(u)$ are linear forms in $u(a), u'(a), \cdots, u^{(n-1)}(a), u(b), u'(b), \cdots, u^{(n-1)}(b)$ and assuming that the homogeneous system $L'(u) = 0$, $U_i'(u) = 0$ is incompatible, he finds that the series set up by the usual method of successive approximations as solutions of the successive equations

$$L'(u_m) = L''(u_{m-1}) + R, \qquad U_i'(u_m) = U_i''(u_{m-1}) + \delta_i$$
$$(m = 1, 2, \cdots)$$

may or may not converge uniformly. The discussion of this question of convergence is made to depend on the characteristic values of the system

$$
(26) \qquad \begin{aligned} L'(u) &= \lambda[L''(u) + R''] + R', \\ U_i'(u) &= \lambda[U_i''(u) + \delta_i''] + \delta_i', \end{aligned}
$$

where $R'' + R' = R$ and $\delta_i'' + \delta_i' = \delta_i$, and (26) reduces to (25) for $\lambda = 1$. If for u_0 we take the solution of $L''(u_0) + R'' = 0$, $U_i'' + \delta_i'' = 0$, the method of successive approximation gives us a power series in λ which will converge within a certain circle (finite or infinite) with $\lambda = 0$ as center and which will diverge without. To obtain a knowledge of the radius of this circle, Bôcher considers a differential system

$$
(27) \qquad L(u) = R, \qquad U_i(u) = \gamma_i \quad (i = 1, 2, \cdots, n),
$$

of which (26) is a special case. The coefficients are assumed continuous in the real variable and analytic in λ for a certain Weierstrass domain. His profound grasp of the algebraic problem leads him to a consideration of the function

$$
(28) \qquad \frac{\begin{vmatrix} u_0 & y_1 & y_2 & \cdots & y_n \\ U_1(u_0) - \gamma_1 & U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ U_n(u_0) - \gamma_n & U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n) \end{vmatrix}}{\begin{vmatrix} U_1(y_1) & \cdots & U_1(y_n) \\ \cdot & \cdot & \cdot \\ U_n(y_1) & \cdots & U_n(y_n) \end{vmatrix}},
$$

where u_1 is a solution of $L(u) = R$ and $y_1, \cdots, y_n$ are a fundamental system of solutions for the homogeneous equation $L(u) = 0$. When λ is not a characteristic number of the homogeneous system $L(u) = 0$, $U_i(u) = 0$, in other words when the denominator is not zero, a substitution shows that (28) is a solution of (27). This solution is continuous in x and analytic in λ except for the characteristic numbers.

But Bôcher is able to proceed further. Even when the denominator of (28) vanishes, as it will for the characteristic number λ_i, it may happen that the numerator has a zero of the same order, in which case a solution of the system (27) still exists for this value of λ. For such a characteristic number both the homogeneous and non-homogeneous systems must have solutions. He had already noted above the corresponding condition of affairs in the algebraic problem, where the solutions of the homogeneous equations (8) are also solutions of the adjoint non-homogeneous equations (9′). This algebraic condition which may be expressed by the formula $\Sigma\delta_j\xi_j = 0$, where ξ_j denotes any solution of (8), has its analogues in the transcendental case.

He is now ready to state the results for (26) and hence for (25). If the circle of convergence of the power series in λ is not infinite, it passes through one of the characteristic values of the homogeneous system

$$L'(u) = \lambda L''(u), \qquad U_i'(u) = \lambda U_i''(u) \tag{29}$$

corresponding to (26). In the ordinary case this characteristic number will be that of smallest absolute value; but if it is not, each solution of (29) corresponding to a λ of smaller absolute value must have index and multiplicity equal, that is, both the homogeneous and non-homogeneous equations have solutions; and at least one solution corresponding to a point on the circle must have them unequal. If the circle of convergence has a radius greater than unity, the approximating series gives a solution of (25). For one-point boundary conditions, such as (24), there are no characteristic numbers and the series converges everywhere.

It may be remarked that the ordinary case of the theorem just given might be conjectured from a consideration of the problem from the standpoint of the integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi)\, u(\xi) d\xi \tag{30}$$

equivalent to (26). By successive substitution of this equation into itself we get the power series

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi) f(\xi) d\xi$$
$$+ \lambda^2 \int_a^b K(x, \xi) \int_a^b K(\xi, \xi_1) f(\xi_1) d\xi_1 d\xi + \cdots.$$

This is the well-known Neumann-Liouville series solution of the integral equation (30), provided it converges uniformly, as it certainly will inside of any circle extending out to the nearest characteristic number of the corresponding homogeneous integral equation. This power series in λ must be identical with that obtained from (26) by successive approximations.

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