CERTAIN PROPERTIES OF BINOMIAL COEFFICIENTS.

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KENYON* has generalized results given by Chrystal† and others for the sum of products of equal powers of the terms of an arithmetic progression and the coefficients of $(x - y)^n$. In evaluating certain integrals connected with the probability curve,‡ the author has had to sum a similar set of products for the coefficients of $(x + y)^{2n}$; each binomial coefficient is multiplied by a given power of the number of that term from the middle term, e.g., the $(n + 1)$th term by 0^q, the nth term by $(-1)^q$, etc., and the results added. One feature of this paper is the distinct gain from the symmetry thus utilized. A recursion formula is obtained for the sums and an expression for the term of highest degree in n, which is all that is needed in the desired application. For odd values of q only the terms from the middle to the end are used.

§1.

The general binomial coefficient of $(x + y)^{2n}$ is $\binom{2n}{n+k}$, where k is the number of the term counting from the middle term, and the sum to be evaluated for even values $q = 2p$ is

$$(1) \qquad \sum_{k=-n}^{n} \binom{2n}{n+k} k^{2p}.$$

It is a familiar theorem that when $p = 0$ this sum is 2^{2n}. It proves that the method of differences can be used here provided that it be applied to the above series divided by 2^{2n}. Let

$$(2) \qquad S(2n, q) \equiv \frac{1}{2^{2n}} \sum_{k=-n}^{n} \binom{2n}{n+k} k^q.$$

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* Kenyon: *Proc. Indiana Academy of Science*, 1914.

† Chrystal: Algebra, Part II, p. 183.

‡ A paper read before the American Mathematical Society, September 5, 1918; this BULLETIN, December, 1919.

On replacing n by $n-1$ the first difference is

$$\sum_{k=-n}^{n}\left[\frac{1}{2^{2n}}\binom{2n}{n+k}-\frac{1}{2^{2n-2}}\binom{2n-2}{n+k-1}\right]k^{2p},$$

(the first and last terms in the second part of the summation being zero); this reduces to

$$\sum_{k=-n}^{n}\frac{1}{2^{2n}}\binom{2n}{n+k}\left[1-\frac{4(n+k)\,(n-k)}{2n(2n-1)}\right]k^{2p}$$

and hence to

$$\frac{2}{n(2n-1)}S(2n,\,2p+2)-\frac{1}{2n-1}S(2n,\,2p).$$

Writing the first difference as

$$D(2n,\,2p)\equiv S(2n,\,2p)-S(2n-2,\,2p),$$

the recursion formula can be written in either of two forms

$$(3)\quad S(2n,\,2p+2)=\frac{n}{2}S(2n,\,2p)+\frac{n}{2}(2n-1)\,D(2n,\,2p),$$

$$(4)\quad S(2n,\,2p+2)=n^2S(2n,\,2p)-\frac{n}{2}(2n-1)\,S(2n-2,\,2p),$$

where in particular by the elementary theorem referred to above

$$(5)\qquad S(2n,\,0)=1.$$

The recursion formula and (5) give the results, after multiplying by 2^{2n}:

$$\sum_{k=-n}^{n}\binom{2n}{n+k}k^2=\frac{n}{2}2^{2n},$$

$$(6)\qquad \sum_{k=-n}^{n}\binom{2n}{n+k}k^4=\frac{n(3n-1)}{4}2^{2n},$$

$$\sum_{k=-n}^{n}\binom{2n}{n+k}k^6=\frac{n(15n^2-15n+4)}{8}2^{2n},$$

. . . .

From these examples it may be inferred that the leading term in $S(2n,\,2p)$ is $1\cdot3\cdot5\cdots(2p-1)\,(n/2)^p$, and this can be proved by using (3) or (4), or as follows. Since $S(2n-2p, 2p)$ is obtained by replacing n in $S(2n, 2p)$ by $n-1$, Maclaurin's theorem gives

$$S(2n-2, 2p) = S(2n, 2p) - S'(2n, 2p) + \tfrac{1}{2}S''(2n, 2p) - \cdots$$

or

$$D(2n, 2p) = S'(2n, 2p) - \frac{1}{2!}S''(2n, 2p) + \cdots,$$

the accents indicating differentiation with respect to n; hence if the leading term of $S(2n, 2p)$ is assumed to be of degree p, that of $D(2n, 2p)$ will be of degree $p-1$ and will be the leading term of $S'(2n, 2p)$. By (3) the leading term of $S(2n, 2p+2)$ is

$$\frac{n}{2}\cdot 1\cdot 3\cdot 5\cdots(2p-1)\left(\frac{n}{2}\right)^{p} + \frac{n}{2}\cdot 2n\cdot 1\cdot 3\cdot 5\cdots(2p-1)\cdot\frac{p}{2}\left(\frac{n}{2}\right)^{p-1}$$

or $1\cdot 3\cdot 5\cdots(2p+1)(n/2)^{p+1}$; whence the theorem follows by mathematical induction. We have thus proved that the leading part of the sum (1) is $1\cdot 3\cdot 5\cdots(2p-1)(n/2)^{p}2^{2n}$, the chief result of this section so far as concerns the properties of binomial coefficients.

With this result the evaluation of the series of integrals dealt with in a previous paper* is exhibited thus, employing the notation used at that time and observing the usual considerations of uniform convergence:

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}x^{2p}\,dx = \lim_{n=\infty}\sum_{k=-n}^{n}\frac{\sqrt{\pi n}}{2^{2n}}\binom{2n}{n+k}(k\Delta x)^{2p}\Delta x$$
$$= \sqrt{\pi}\lim_{n=\infty}\sqrt{n}S(2n, 2p)\cdot\frac{\sigma^{2p+1}}{(n/2)^{(2p+1)/2}}$$
$$= \sigma^{2p+1}\sqrt{2\pi}\,1\cdot 3\cdot 5\cdot\cdots(2p-1).$$

In the language of statistics one may from this say, for example, that the mean square deviation of the area under the curve $y = e^{-x^2/2\sigma^2}$ is σ^2; in other nomenclature we may say that the second, fourth, $\cdots$ moments of this area are σ^2, σ^4, $\cdots$. It is believed that the method of evaluating this already known integral is new.

* L. c.

§ 2.

For the case where $q = 2p + 1$, we sum in like manner

$$(7) \qquad R(2n, 2p+1) \equiv \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{n+k} k^{2p+1},$$

inasmuch as considerations of symmetry show $S(2n, 2p + 1)$ to be zero. The method of § 1 gives the recursion formula

$$(8) \quad R(2n, 2p+3) = \frac{n}{2}(2n, 2p+1) + \frac{n}{2}(2n-1)\bar{D}(2n, 2p+1),$$

where

$$\bar{D}(2n, 2p+1) \equiv R(2n, 2p+1) - R(2n-2, 2p+1).$$

Now

$$R(2n, 1) \equiv \frac{1}{2^{2n}}\left[0 \cdot \binom{2n}{n} + 1 \cdot \binom{2n}{n-1} + \cdots \right.$$
$$\left. + (n-2) \cdot \frac{2n\,(2n-1)}{2!} + (n-1) \cdot 2n + n \cdot 1 \right].$$

The sum of the last two terms enclosed in brackets is $n(2n-1)$, the sum of the last three terms is $\frac{n(2n-1)\,(2n-2)}{2!}$, etc., and mathematical induction proves that the sum of all these terms is $\frac{n}{2}\binom{2n}{n}$, i.e., $n/2$ times the first coefficient. Thus,

$$(9) \qquad R(2n, 1) = \frac{1}{2^{2n}} \frac{n}{2} \binom{2n}{n}.$$

Formulas (8) and (9) enable us to write the successive values

$$\sum_{k=0}^{n} \binom{2n}{n+k} k = \frac{n}{2}\binom{2n}{n},$$

$$\sum_{k=0}^{n} \binom{2n}{n+k} k^3 = 2\left(\frac{n}{2}\right)^2 \binom{2n}{n},$$

$$\sum_{k=0}^{n} \binom{2n}{n+k} k^5 = 2\left(\frac{n}{2}\right)^2 (2n-1) \binom{2n}{n},$$

$$\sum_{k=0}^{n} \binom{2n}{n+k} k^7 = 2\left(\frac{n}{2}\right)^2 (6n^2 - 8n + 3) \binom{2n}{n},$$

. . . .

By mathematical induction it is proved that the leading term in

$$\sum_{k=0}^{n} \binom{2n}{n+k} k^{2p+1}$$

is

$$\frac{1}{2} p!\, n^{p+1} \binom{2n}{n}.$$

Corresponding to the application made at the end of § 1, we have here

$$\int_0^\infty e^{-x^2/2\sigma^2} x^{2p+1} dx = \lim_{n=\infty} \frac{1}{\binom{2n}{n}} \sum_{k=0}^{\infty} \binom{2n}{n+k} (k\Delta x)^{2p+1} \Delta x$$

$$= \lim_{n=\infty} \frac{1}{2} p!\, n^{p+1} \left(\frac{2\sigma^2}{n}\right)^{p+1}$$

$$= \frac{1}{2} p!\, (2\sigma^2)^{p+1}.$$

For example, since the area under the whole curve $y = e^{-x^2/2\sigma^2}$ is $\sigma \sqrt{2\pi}$, the "mean deviation" of this area is $\sigma \sqrt{2/\pi}$.

The products of the binomial coefficients by powers of terms of other arithmetical progressions do not seem to give simple results analogous to those obtained by Kenyon; this question is reserved for further study.

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THE WORK OF POINCARÉ ON AUTOMORPHIC FUNCTIONS.

Oeuvres de Henri Poincaré, publiées sous les auspices du Ministère de l'Instruction publique par G. DARBOUX. Tome II, publié avec la collaboration de N. E. NÖRLUND et de ERNEST LEBON. Paris, Gauthier-Villars, 1916. lxxi + 632 pp.

THE collected works of Poincaré will fill some 10 volumes, of which the one before us is the first to be published. It contains the principal papers written by him in the field of