are satisfied, the inequalities

$$|g_n(x) - g_n(\xi_i)| \leqq \max|f| \cdot V_y[K(x, y) - K(\xi_i, y)]$$
$$(n = 1, 2, \cdots),$$

and inequality (7) will imply that

$$|g_n(x) - g_{n+m}(x)| \leqq 2[\max|f| + 1]\epsilon$$
$$(n \geqq N'; \; m = 1, 2, \cdots).$$

Consequently these g's must be uniformly convergent, and the transformation must be completely continuous.

If condition B is not satisfied there must be an $\epsilon > 0$ and an infinite sequence of x's such that

$$V_y[K(x_m, y) - K(x_n, y)] \geqq \epsilon \qquad (m \neq n).$$

It follows from the principal theorem of the first section, that if V_yK is bounded uniformly, a subset of these x's must make $K(x_n, y)$ convergent. As no subset of them can satisfy equation (4), condition A cannot be satisfied. Therefore when A is satisfied B must be, and since A is necessary and B sufficient, B must be necessary and A sufficient also.

TRINITY COLLEGE,
    HARTFORD, CONN.

ON THE RELATION OF THE ROOTS AND POLES OF A RATIONAL FUNCTION TO THE ROOTS OF ITS DERIVATIVE.

BY MR. BEN-ZION LINFIELD.

1. F. Lucas, *Journal de l'Ecole polytechnique*, 1879, gave a mechanical proof of the following theorem:

The roots of the derivative of a cubic are the foci of the maximum ellipse inscribed in the triangle whose vertices are the roots of the cubic.

Professor Maxime Bôcher gave a simple proof of this theorem, *Annals of Mathematics*, volume 7, page 70, 1892. Here he made observations on the general theorem concerning the polynomial of nth degree and asked the question, "Could

not the first of these propositions be brought into connection with the focal properties of the higher plane curves? " Other demonstrations of Lucas's theorem have been given, but the writer is not aware of Professor Bôcher's question having been answered. It is the object of this note to answer this question by the following:

2. THEOREM. *The roots of the derivative of the rational function*

$$f(z) \equiv \prod_{i=1}^{n} (z - z_i)^{\mu_i}$$

are the multiple roots of $f(z)$ to one lower order and the foci of a curve of class $n - 1$ which touches each segment $z_i z_j$ $(i \neq j,$ $= 1, 2, \cdots, n)$ at a point dividing it in the ratio of μ_i to μ_j.

3. Let

$$\alpha_i \equiv u x_i + v y_i + w \qquad (i = 1, 2, \cdots, n)$$

be the line equation of the point $P_i \equiv (x_i, y_i)$, where $z_i \equiv x_i + y_i \sqrt{-1}$ are the roots and poles of the rational function $f(z)$. Then,

$$(1) \qquad\qquad \Pi \equiv \alpha_1 \alpha_2 \cdots \alpha_n = 0$$

is the line equation of the n points x_i, y_i.

$$(2) \qquad \varphi(u, v, w) \equiv \sum_{i=1}^{n} \mu_i \frac{\partial \Pi}{\partial \alpha_i} = 0$$

is the tangential (line) equation of a curve of class $n - 1$ tangent to the segments $P_i P_j$.

The point of contact of an arbitrary tangent to (2) whose coordinates are u, v, w, is given by equations

$$\frac{X}{\dfrac{\partial \varphi}{\partial u}} = \frac{Y}{\dfrac{\partial \varphi}{\partial v}} = \frac{1}{\dfrac{\partial \varphi}{\partial w}}.$$

Also from (2)

$$\frac{\partial \varphi}{\partial u} = \Sigma(\mu_j x_i + \mu_i x_j) \frac{\partial^2 \Pi}{\partial \alpha_i \partial \alpha_j},$$

$$\frac{\partial \varphi}{\partial v} = \Sigma(\mu_j y_i + \mu_i y_j) \frac{\partial^2 \Pi}{\partial \alpha_i \partial \alpha_j},$$

$$\frac{\partial \varphi}{\partial w} = \Sigma(\mu_i + \mu_j) \frac{\partial^2 \Pi}{\partial \alpha_i \partial \alpha_j}.$$

Therefore the coordinates of the point of contact of the tangent (u, v, w) with (2) are

$$X = \frac{\Sigma(\mu_j x_i + \mu_i x_j)\partial^2\Pi/\partial\alpha_i\partial\alpha_j}{\Sigma(\mu_i + \mu_j)\partial^2\Pi/\partial\alpha_i\partial\alpha_j},$$

$$Y = \frac{\Sigma(\mu_j y_i + \mu_i y_j)\partial^2\Pi/\partial\alpha_i\partial\alpha_j}{\Sigma(\mu_i + \mu_j)\partial^2\Pi/\partial\alpha_i\partial\alpha_j}.$$

Let the tangent (u, v, w) coincide with the tangent through the two points P_p and P_q. Then

$$\frac{\partial^2\Pi}{\partial\alpha_i\partial\alpha_j} = 0 \qquad \text{(when } i \neq p,\ j \neq q\text{)},$$

since $\alpha_p = 0$, $\alpha_q = 0$ and α_p or α_q occurs in each of these second derivatives except $\partial^2\Pi/\partial\alpha_p\partial\alpha_q$. Hence the coordinates of the point of contact of (2) with P_pP_q are

$$X = \frac{\mu_q x_p + \mu_p x_q}{\mu_q + \mu_p}, \qquad Y = \frac{\mu_q y_p + \mu_p y_q}{\mu_q + \mu_p},$$

dividing the segment P_pP_q in the ratio $\mu_p : \mu_q$.

4. The foci of the curve $\varphi(u, v, w) = 0$ are the roots of the polynomial $\varphi(-1, -\sqrt{-1}, z)$* where $z \equiv x + y\sqrt{-1}$. Thus in (2)

$$\alpha_i \equiv z - z_i, \qquad \Pi \equiv \prod_{i=1}^{n}(z - z_i).$$

Also

$$\varphi(-1, -\sqrt{-1}, z) \equiv \sum_{j=1}^{n}\mu_j\frac{\prod_{i=1}^{n}(z - z_i)}{z - z_i}.$$

But the derivative of $f(z)$ can be written

$$f'(z) = \prod_{i=1}^{n}(z - z_i)^{\mu_i - 1} \cdot \sum\mu_j\frac{\prod_{i=1}^{n}(z - z_i)}{z - z_j},$$

which establishes the theorem, § 2, as enunciated.

5. The present note is an abstract presenting the central theorem of a paper wherein the general curve $\varphi = 0$ of class

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* Salmon, Higher Plane Curves, 3d ed., § 141; also Emch, this BULLETIN, vol. 25, pp. 157–161.

$n - 1$ is discussed and methods evolved for drawing it. We go no further here than to notice the interesting case when $n = 3$, of which the theorem of Lucas is a particular case.

6. When $n = 3$ the roots of the derivative of

$$f(z) \equiv \prod_{i=1}^{3} (z - z_i)^{\mu_i}$$

are the roots of $\prod_{1}^{3} (z - z_i)^{\mu_i - 1}$ and the foci of

$$\varphi \equiv \mu_1 \alpha_2 \alpha_3 + \mu_2 \alpha_1 \alpha_3 + \mu_3 \alpha_1 \alpha_2 = 0.$$

The curve is clearly of the second class and therefore a conic. The center z_c is given by

$$\frac{\partial}{\partial z} \Sigma \mu_j \frac{\Pi(z - z_i)}{z - z_j} = 0,$$

or

$$z_c = \frac{(\mu_2 + \mu_3)z_1 + (\mu_3 + \mu_1)z_2 + (\mu_1 + \mu_2)z_3}{2(\mu_1 + \mu_2 + \mu_3)}.$$

When the powers (μ) are positive the conic $\varphi = 0$ touches the sides of the triangle z_1, z_2, z_3 internally, and is therefore an ellipse inscribed in the triangle; the point of contact divides each side in the ratio of the powers of the vertices.   If $\mu_1 = \mu_2 = \mu_3 = 1$, $f(z)$ becomes a cubic and the conic touches the sides of the triangle at their midpoints, which is Lucas's case.

When each power is not positive, it is only necessary to consider one as being negative, for changing the signs of all the powers does not effect $\varphi = 0$.

If $\mu_1 + \mu_2 + \mu_3 = 0$, the conic is a parabola, the center being at infinity. It follows from a well-known theorem, the focus of a parabola touching the three sides of a triangle lies on the circumcircle of the triangle, that in this case the roots of the derivative of $f(z)$ lie on a circle through the roots of $f(z)$.

A conic is clearly an ellipse or hyperbola according as it passes through a point between or not between two parallel tangents, respectively.

The tangents parallel to the sides of the triangle are easily found, since the center is known and the point of touch on a

side. Let the side z_1z_2 be the x-axis. The equation of the tangent parallel to this axis is

$$y = \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 + \mu_3} y_3,$$

with similar equations of the tangents parallel to the other two sides. Let μ_3 be negative. The points of contact with the sides z_1z_3 and z_2z_3 lie between the x-axis and the parallel tangent or outside them according as

$$\mu_1 + \mu_2 + \mu_3 \gtreqless 0,$$

and the conic is accordingly an ellipse or hyperbola.

Hence if all the powers be positive $\varphi = 0$ is an ellipse. If one of the powers be negative it is an ellipse, parabola or hyperbola according as their sum is less than, equal to or greater than zero. If two of the powers be negative it is an ellipse, parabola or hyperbola according as their sum is greater than, equal to or less than zero. In particular if $\mu_1 + \mu_3 = 0$ the side z_1z_3 is an asymptote. If $\mu_1 = \mu_2 = -\mu_3$ the sides z_1z_3 and z_2z_3 are asymptotes. If in addition the triangle $z_1z_2z_3$ is isosceles having its vertex at z_3, the circle having z_3 for its center and passing through z_1 and z_2 passes through the foci of $\varphi = 0$, for the altitude and half base of this triangle are the major and minor axes respectively.

UNIVERSITY OF VIRGINIA,
　　May 1, 1919.

MORITZ CANTOR, THE HISTORIAN OF MATHEMATICS.

BY PROFESSOR FLORIAN CAJORI.

PROFESSOR Moritz Cantor died at Heidelberg on April 10, 1920, in his ninety-first year, three days after the death of his only son. Considering his own long career and that of many other mathematicians, he could well remark on his eightieth birthday, "Die Mathematik gehört nicht gerade zu den ungesunden Handwerken."[*] His literary activity ex-

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[*] Felix Müller, Der Mathematische Sternenhimmel des Jahres 1811, Leipzig und Berlin, 1911, p. 7.