BROCARD: 24 propositions de Fermat.
DELAPORTE: Sur la réforme du calendrier.
DU PASQUIER: Sur les nombres transfinis.
GREENHILL: Les fonctions de Fourier et Bessel comparées.
D'OCAGNE: La pratique courante de la méthode nomographique des points alignés: à propos de ses applications de guerre.
GROSSMANN: Sur l'état de publication des œuvres d'Euler.
POSTIGLIONE: Cyclómetrie mécanique.
DUBECQ: Communication sur l'enseignement, République Argentine.
ZERVOS: Sur l'enseignement mathématique.

DAVID EUGENE SMITH.

NOTE ON A METHOD OF PROOF IN THE THEORY OF FOURIER'S SERIES.

BY PROFESSOR DUNHAM JACKSON.

(Read before the American Mathematical Society September 7, 1920.)

IT has been pointed out on various occasions* that if $f(x)$ is a continuous function of period 2π satisfying the Lipschitz-Dini condition, that is, if $\lim_{\delta=0}\omega(\delta)\log\delta = 0$, where $\omega(\delta)$ is the maximum of the oscillation of $f(x)$ in an interval of length δ, then the uniform convergence of the Fourier series for $f(x)$ can be inferred almost immediately from the following two propositions:

A.† If $f(x)$ satisfies the Lipschitz-Dini condition,‡ there exists for every positive integral value of n a finite trigonometric sum $\tau_n(x)$, of order n at most, such that $\lim_{n=\infty} r_n \log n = 0$, where r_n is the maximum of $|f(x) - \tau_n(x)|$.

---

* Cf., e.g., Lebesgue, "Sur les intégrales singulières," *Annales de la Faculté de Toulouse*, series 3, vol. 1 (1909), pp. 25–117; pp. 116–117.

† Cf., e.g., Lebesgue, loc. cit., p. 116; D. Jackson, "On the approximate representation of an indefinite integral, etc.," *Transactions Amer. Math. Society*, vol. 14 (1913), pp. 343–364; p. 350.

‡ It is understood throughout the paper that every function considered has the period 2π.

*B.** If $\varphi(x)$ is any continuous† function, and $S_n(x)$ the partial sum of the Fourier series for $\varphi(x)$ to terms of order n, then $|S_n(x)|$ can not exceed $KM \log n$, where M is the maximum of $|\varphi(x)|$, and K is an absolute constant.

The central point in the proof is the fact that $\tau_n(x)$ is identical with the partial sum of its own Fourier series to terms of order n. It is the purpose of this note to show that similar reasoning can be applied to the arithmetical mean of Fejér,‡ in spite of the fact that the Fejér mean formed for a finite trigonometric sum $\tau_n(x)$ is not the same as $\tau_n(x)$. It is necessary to change the argument somewhat, but there is no difficulty in making the required modification.

Let $f(x)$ be now an arbitrary continuous function, and let $\sigma_n(x)$ be the arithmetical mean of the partial sums of the Fourier series for $f(x)$, to terms of order n. The uniform convergence of $\sigma_n(x)$ to the value $f(x)$ is to be deduced from the propositions:

C. (Weierstrass's theorem.)§ If $f(x)$ is continuous, there exists for every positive integral value of n a finite trigonometric sum $\tau_n(x)$, of order n at most, such that $\lim_{n=\infty} r_n = 0$, where r_n is the maximum of $|f(x) - \tau_n(x)|$.

D.‖ If $\varphi(x)$ is any continuous function (more generally, any integrable function), and $\sigma_n(x)$ the Fejér mean of the Fourier series for $\varphi(x)$ to terms of order n, then $|\sigma_n(x)|$ can not exceed M, where M is the maximum of $|\varphi(x)|$.

Let ϵ be any positive quantity. Let a finite trigonometric sum $\tau_p(x)$, of order p, be determined, according to Proposition C, so that

$$|f(x) - \tau_p(x)| < \tfrac{1}{3}\epsilon.$$

Let $\sigma_{n1}(x)$ be the Fejér mean, of order n, for the function $\tau_p(x)$, and $\sigma_{n2}(x)$ the corresponding mean for the function $f(x) - \tau_p(x)$.

* Cf., e.g., Lebesgue, loc. cit., p. 116; D. Jackson, "On approximation by trigonometric sums and polynomials," *Transactions Amer. Math. Society*, vol. 13 (1912), pp. 491–515; pp. 502, 512–515.

† It is sufficient for the truth of the statement that φ be integrable, but for present purposes there is no need of speaking of any but continuous functions.

‡ Fejér, "Untersuchungen über Fouriersche Reihen," *Mathematische Annalen*, vol. 58 (1904), pp. 51–69.

§ Weierstrass, "Ueber die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen," *Berliner Sitzungsberichte*, 1885, pp. 633–639, 789–805; p. 801.

‖ Fejér, loc. cit., p. 60.

Then

$$\sigma_n(x) = \sigma_{n1}(x) + \sigma_{n2}(x). \tag{1}$$

By Proposition D,

$$|\sigma_{n2}(x)| < \tfrac{1}{3}\epsilon,$$

$$|f(x) - \tau_p(x) - \sigma_{n2}(x)| < \tfrac{2}{3}\epsilon, \tag{2}$$

for all values of n. The quantity $\sigma_{n1}(x)$ is the arithmetical mean of $n + 1$ finite trigonometric sums, of which all from the $(p + 1)$th on, if $n \geqq p$, are identical with $\tau_p(x)$, while each of the first p is composed of a part of the terms of $\tau_p(x)$. Added together, the first p sums which enter into the mean give a finite trigonometric sum $\omega_{p-1}(x)$, which is of order $p - 1$ at most, and independent of n. So $\sigma_{n1}(x)$ can be written in the form

$$\sigma_{n1}(x) = \frac{\omega_{p-1}(x) + (n + 1 - p)\tau_p(x)}{n + 1}$$

$$= \tau_p(x) + \frac{\omega_{p-1}(x) - p\tau_p(x)}{n + 1}.$$

As the last numerator is independent of n, $\sigma_{n1}(x)$ approaches $\tau_p(x)$ uniformly as n becomes infinite—a fact which is fairly obvious in the first place—and, if n is sufficiently large,

$$|\tau_p(x) - \sigma_{n1}(x)| < \tfrac{1}{3}\epsilon. \tag{3}$$

By combination of (1), (2), and (3), for values of n satisfying (3),

$$|f(x) - \sigma_n(x)| < \epsilon,$$

which completes the convergence proof.

THE UNIVERSITY OF MINNESOTA,
MINNEAPOLIS, MINN.