ON IMPLICIT FUNCTIONS.

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METHODS of solving a system of m equations in $n > m$ variables, by reducing the problem to that of solving a set of differential equations, have already been given;* the aim of this paper is to present a method by which the system of m equations can be reduced immediately to a system of m differential equations of the first order, from which reduction the existence theorem and a method of constructing the solutions follow directly. For simplicity the case of two equations in four variables will be treated.

§ 1. *Reduction to a System of Differential Equations.*

It is required to solve the system

$$f_1(x, y, u, v) = 0, \quad f_2(x, y, u, v) = 0 \tag{1}$$

in the neighborhood of a set of values (x_0, y_0, u_0, v_0) for which

$$f_1(x_0, y_0, u_0, v_0) = 0, \quad f_2(x_0, y_0, u_0, v_0) = 0.$$

There is no loss in generality in assuming $x_0 = y_0 = u_0 = v_0 = 0$; suppose x, y to be the independent, u, v the dependent variables. It will be assumed that all the first partial derivatives exist and are continuous in the neighborhood of the origin defined by the inequalities

$$\sqrt{x^2 + y^2} \leqq a, \quad |u| \leqq b, \quad |v| \leqq b. \tag{R}$$

Also, assume that the Jacobian

$$\Delta = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}$$

does not vanish at the origin (0, 0, 0, 0); consequently the constants a, b can be so chosen that $\Delta \neq 0$ in R. Introduce

* Horn, Gewöhnliche Differentialgleichungen beliebiger Ordnung.

polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Equations (1) become

$$(1') \quad f_1(r\cos\theta, r\sin\theta, u, v) = 0, \quad f_2(r\cos\theta, r\sin\theta, u, v) = 0.$$

If functions $u(r, \theta)$, $v(r, \theta)$, satisfying (1′) can be found, such that $\partial u/\partial r$, $\partial v/\partial r$ are defined throughout the region R, we must have

$$(2) \quad \begin{aligned} &\frac{\partial f_1}{\partial x}\cos\theta + \frac{\partial f_1}{\partial y}\sin\theta + \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial f_1}{\partial v}\frac{\partial v}{\partial r} = 0, \\ &\frac{\partial f_2}{\partial x}\cos\theta + \frac{\partial f_2}{\partial y}\sin\theta + \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial f_2}{\partial v}\frac{\partial v}{\partial r} = 0. \end{aligned}$$

Since $\Delta \neq 0$,

$$(3) \quad \begin{aligned} \frac{\partial u}{\partial r} &= \frac{\begin{vmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial x} \end{vmatrix}\cos\theta + \begin{vmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial y} \end{vmatrix}\sin\theta}{\Delta}, \quad u(0) = 0, \\ \frac{\partial v}{\partial r} &= \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial u} \end{vmatrix}\cos\theta + \begin{vmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial u} \end{vmatrix}\sin\theta}{\Delta}, \quad v(0) = 0. \end{aligned}$$

In these equations consider θ as a parameter, u, v, r as the variables; equations (3) can be written in the form

$$(3') \quad \frac{du}{dr} = A(u, v, r), \quad \frac{dv}{dr} = B(u, v, r), \quad u(0) = v(0) = 0.$$

Suppose the original functions f_1, f_2 such that for every θ, $0 < \theta \leqq 2\pi$, there exist constants C, D if u', u'', v', v'', are in R, such that

$$(4) \quad \begin{aligned} &|A(u'', v'', r) - A(u', v', r)| < C|u'' - u'| + D|v'' - v'|, \\ &|B(u'', v'', r) - B(u', v', r)| < C|u'' - u'| + D|v'' - v'|. \end{aligned}$$

§ 2. *Final Solution.*

Suppose that for $u = v = 0$, $r \leqq h$, $|A| \leqq M_0$, $|B| \leqq M_0$; for u, v, r in R, $|A| \leqq M$, $|B| \leqq M$. Then from the existence theorem for ordinary differential equations,* if $C + D = K$, the equations (3′) have solutions u, v if $0 \leqq r \leqq \delta$, where δ is the larger of the quantities δ_1, δ_2,

$$\delta_1 \leqq a \leqq \frac{b}{M}, \quad \delta_2 \leqq a \leqq \frac{1}{K}\log\left(1 + \frac{bK}{M_0}\right).$$

The functions u, v satisfy (3′), hence have derivatives with respect to r.

Since A, B are periodic in θ, the constants C, D have upper limits independent of θ, consequently K, M_0 can be defined independent of θ; the series defining u, v converge uniformly in θ, hence u, v are continuous functions of θ. The constant M can also be defined independent of θ, with the result that the solutions u, v are continuous functions of r, θ, and differentiable with respect to r, if $0 \leqq r \leqq \delta$.

Introduce these functions in (1′); since

$$\frac{\partial f_1}{\partial r} = \frac{\partial f_2}{\partial r} = 0, \quad f_1 = \phi_1(\theta), \quad f_2 = \phi_2(\theta).$$

But within the region $0 \leqq r \leqq \delta$, f_1, f_2 are continuous functions of r, θ; for $r = 0$, $x = y = u = v = 0$, consequently

$$\phi_1(\theta) = 0, \quad \phi_2(\theta) = 0.$$

These equations must hold for every θ, hence are identities, and equations (1′) are identically satisfied. The functions u, v are the solutions required.

For n variables $x_1, x_2, \cdots, x_n$,

$$r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}, \qquad \cos\theta_i = x_i/r.$$

* Picard, Traité d'Analyse, 2, pp. 343, 345.