ON A PENCIL OF NODAL CUBICS. SECOND PAPER.

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1. REFERRING to the notations of my first communication,* let Γ_n, Γ_s be two cubics of the pencil Γ, and let X_n, X_s, F be the points of intersection of a fixed line l through the double point O with Γ_n, Γ_s, and the basic line ABC respectively; let X_n', X_s', F' be the analogous points on any other line l' through O. The lines X_nX_n', X_sX_s', meet on ABC according to proposition 8 of the paper referred to, and therefore

$$(OFX_nX_s) = (OF'X_n'X_s').$$

Thus a variable line l' through the double point O meets the two cubics Γ_n, Γ_s in two points X_n', X_s' which, with the double point O and the trace F' of l' on ABC, form an anharmonic ratio having a constant value. Consequently:†

Two nodal cubics having in common three collinear points, the double point, and the tangents at this point, are homological.

The double point and the base of the three collinear points are respectively the center and the axis of homology.

Two tricuspidal quartics having in common three concurrent tangents, the double tangent, and the two points of contact with this line, are homological.

The double tangent and the common point of the three concurrent tangents are respectively the axis and the center of homology.

2. Any two cubics of the pencil Γ are thus homological, i.e., Γ is a pencil of homological cubics. This fundamental property of Γ furnishes immediately a second proof of most of the propositions of my first paper. It also brings to light many new propositions. For instance:

* This BULLETIN, vol. 26, p. 203 (February, 1920).

† M. Chasles, Traité de Géométrie supérieure, 2d edition, p. 350.

(*a*) The triangle formed by the three inflexional tangents of any cubic of the pencil Γ is homological to the corresponding triangle of any other cubic of the pencil, the double point O being the center of homology, hence: *The vertices of the triangle formed by the inflexional tangents of a variable cubic of the pencil* Γ *describe three straight lines passing through the double point.*

(*b*) It is known that the lines joining the pairs of corresponding points on a nodal cubic Γ_n envelop a conic ω_n, which is the Cayleyan of the cubic of which Γ_n is the Hessian. The conics (ω) corresponding to ω_n in the various curves of the pencil Γ are homological to ω_n. Hence: *The conics* (ω) *of the cubics of the pencil* Γ *have two points in common on the basic line of the pencil.*

It should be noticed however that the conics (ω) do not, in general, form a pencil. Being homological, in order to form a pencil, they would have to be tangent to the lines projecting from the center of homology O the two points that the conics have in common with the axis of homology ABC* at these points. Now it is known that the tangents from O to ω_n are the tangents OT_1, OT_2 to Γ_n at O, their points of contact with ω_n being the points determined on OT_1, OT_2 by the inflectional line i_n of Γ_n.

3. In the cases when the common elements of the two curves of section 1 are of some special nature or are taken in some special position the mutual relation of the two curves may become of particular interest. For lack of space we shall give only one example.

The tricuspidal hypocycloid (or tricusp) is a quartic touching the line at infinity at the cyclical points, according to a proposition due to Cremona.† If we consider two such curves having three concurrent tangents in common, their axis of homology will be at infinity. Consequently: *Two tricuspidal hypocycloids having three concurrent tangents in common are similar and similarly placed. The point of intersection of their common tangents is the center of similitude of the two curves.*

University of Oklahoma,
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* C. Servais, "Sur les faisceaux de coniques," sec. 9, *Le Matematiche pure ed applicate*, vol. I (December, 1901).

† See R. C. Archibald, "The cardioid and the quartics with three cusps," *Annals of Mathematics*, 2d series, vol. 4 (1903).