may learn a good deal about the properties of algebraic curves, so that in this respect the publication of a new English treatise on curves is not without value, and deserves commendation.

ARNOLD EMCH.

Nouvelles Méthodes de Résolution des Equations du 3e Degré. By le VTE. DE GALEMBERT. Paris, Vuibert, 1919. 22 pp.

This pamphlet gives a method for rapid numerical calculation of the real roots of the cubic equation

$$x^3 + Ax^2 + Bx + C = 0,$$

new as far as I am aware, for the case in which there are three real roots. The equation is reduced to the form $z^3 - 3z = y$; the latter equation has three real roots if $|y| \leqq 2$. A table of corresponding values of z and y is computed once for all, by means of which the values of z may be found accurately to two decimals, whenever y is known to six places. An additional table gives z and y in terms of x, A, B and C, for the different combinations of signs which the coefficients may have in the general equation. The actual calculation of the roots is very much simplified in this way.

A. DRESDEN.

The Integral Calculus. By JAMES BALLANTYNE. Boston, Published by the author, 1919. 41 pp.

THE subtitle of this small book is sufficiently descriptive of its scope. It reads, "On the integration of the powers of transcendental functions, new methods and theorems, calculation of Bernouillian numbers, rectification of the logarithmic curve, integration of logarithmic binomials, etc."

The author has several new series expansions of transcendental functions; but does not burden his tale with arguments as to the rigor of his methods or the general validity of his formal results. The book gives the impression of having been written for the fun of it, by a very ingenious gentleman, who was having a fine time giving free rein to his analytical processes and going gladly wherever those steeds—dangerous if unchecked—might lead him.

The style of the text may be indicated by such expressions of olden time flavor as, "integrals of even powers of sin xdx to radius 1"; "the value of C is the area of the full quadrant of the curvilinear"; "the two curves $y = 1/\sin^m x$ and $y = 1/\cos^m x$ having their origin at opposite ends of the axis of $x \cdots$"; "when m is an odd negative, there will always appear in the series one irrational term, namely $\pm \frac{1}{0} \tan^0 x$"; "that is, the logarithm of an infinite number, multiplied by zero = nothing."

In section 1 various reduction formulas for integrals of $\sin^m x$, $\cos^m x$, etc., are developed and also formulas expressing those integrals in series of trigonometric functions, e.g.: "The integral of $\sin^m x dx$ for any value of m is

$$A(1 - \cos x) - \frac{B}{3}(1 - \cos^3 x) + \frac{C}{5}(1 - \cos^5 x) - \frac{D}{7}(1 - \cos^7 x) + \cdots,$$

where $A, B, C, D, \cdots$ are the successive terms in the development of the binomial $(1 + 1)^{(m-1)/2}$." These finite and series developments are equated in section 2 to derive some well known and new formulas. For example, if we let $x = \frac{1}{2}\pi$ in the above series, for $m = 0$, and equate it to $\int \sin^0 x dx = x$, we obtain

$$1 + \frac{1}{2\cdot 3} + \frac{3}{2\cdot 4\cdot 5} + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 7} + \cdots = \pi/2.$$

Again by equating $\int \cos^0 x dx = x$ to

$$\int \cos^m x dx = A \sin x - \frac{B}{3}\sin^3 x + \frac{C}{5}\sin^5 x \cdots,$$

we have

$$x = \sin x + \frac{1}{2\cdot 3}\sin^3 x + \frac{3}{2\cdot 4\cdot 5}\sin^5 x \cdots$$

"an inversion of the series given by Newton for the sine in terms of the arc."

From the series expansion, for m odd,

$$\int_0^{\pi/2} \sin^m x dx = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdots\frac{m-1}{m}$$

and from the finite expansion, for m even,

$$\int_0^{\pi/2} \sin^m x dx = \frac{\pi}{2}\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdots\frac{m-1}{m}\right)$$

the author concludes, "But if m be considered infinite, the distinction between odd and even values of m vanishes and these two expressions will be equal to each other." It follows that

$$\frac{\pi}{2}=\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9}\cdots,$$

the expression given by Wallis.

Section 3 takes up the series for expressing the tangent in terms of the arc and its relation to the Bernoullian numbers. From $\int \tan x dx = -\log \cos x$, the author gets

$$\tan x = x + \frac{2}{2\cdot 3}x^3 + \frac{16}{2\cdot 3\cdot 4\cdot 5}x^5 + \frac{272}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \cdots$$

and works out a scheme for finding the numerators N_a of the terms of this expansion, namely,

$$N_a = \pm 1 \mp aN_1 \pm \frac{a(a-1)(a-2)}{2\cdot 3}N_3 \mp \frac{a(a-1)(a-2)(a-3)(a-4)}{2\cdot 3\cdot 4\cdot 5}N_5 \pm \cdots.$$

to the term N_{a-2}, where a is the index of the power of x whose coefficient is required and using the upper or lower sign according as $\frac{1}{2}(n-1)$ is even or odd. The author concludes this section with a practical method for the calculation of the Bernoullian numbers. The derivation of the formula is by the theory of differences, but it gives the numbers with more facility than the usual formulas. This formula is

$$1 = \frac{1}{P+1} + \frac{1}{2} + \frac{P}{2}B_1 - \frac{P(P-1)(P-2)}{2\cdot 3\cdot 4}B_2 + \frac{P(P-1)(P-2)(P-3)(P-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}B_3 - \cdots,$$

a series terminating at $B_{P/2}$. By assuming positive, even values for P the numbers B_i may be read off.

In section 4 integrals of powers of transcendental functions in terms of these functions are given. The procedure is as follows: Let $y = \varphi(x)$. Express dy/dx in terms of y and integrate $y^m/(dy/dx)$ with respect to y (usually in series form) "as an independent variable and as an algebraic function." The result will be the integral of $\varphi^m(x)dx$ expressed in terms of the function itself, whether it be algebraic or transcendental. The formulas derived at this point are of immediate applicability to the quadrature and rectification of, for example, such troublesome curves as the logarithm.

The final section on "the integration of the logarithm of binomials and other complex quantities" is less satisfactory and of less interest than the earlier sections and is made up of various odds and ends.

E. GORDON BILL.

NOTES.

AT the meeting of the National Academy of Sciences held at Princeton University November 16–17, 1920, the following mathematical papers were read: By E. B. WILSON : "Equipartition of energy"; by EDWARD KASNER: "Einstein gravitational fields: orbits and light rays"; by J. W. ALEXANDER: "Knots and Riemann spaces"; by PHILIP FRANKLIN: "The map coloring problem."

THE July number (volume 42, number 3) of the *American Journal of Mathematics* contains the following papers: "The failure of the Clifford chain," by W. B. CARVER; "On the representations of numbers as sums of 3, 5, 7, 9, 11 and 13 squares," by E. T. BELL; "On a certain class of rational ruled surfaces," by ARNOLD EMCH.

THE opening (September) number of volume 22 of the *Annals of Mathematics* contains: "On multiform functions defined by differential equations of the first order," by PIERRE BOUTROUX; "Hermitian metrics," by J. L. COOLIDGE; "On the expansion of certain analytic functions in series," by R. D. CARMICHAEL; "Notes on the cyclic quadrilateral," by F. V. MORLEY: "Note on the preceding paper," by F. MORLEY; "Qualitative properties of the ballistic trajectory," by T. H. GRONWALL.