This is the surface of revolution of a parabola of latus rectum $8m$ about its directrix. A similar result was obtained by Flamm* who considered the surface, in euclidean three-space, for which the linear element is given by (2) for $u_2 = \pi/2$.

PRINCETON UNIVERSITY,
April 16, 1921.

A COVARIANT OF THREE CIRCLES.

BY PROFESSOR A. B. COBLE.

(Read before the American Mathematical Society April 23, 1921.)

Dr. J. L. Walsh † has stated the following theorem.

THEOREM. *If the double ratio,* $(z_1, z_3 \mid z_2, z)$, *of the four points* z_1, z_2, z_3, z *in the complex plane is a real number* λ, *then as the points* z_1, z_2, z_3 *run over the circles* C_1, C_2, C_3 *(and their interiors) respectively, the locus of* z *is a circle (and its interior).*

This locus is evidently a covariant, under the inversive group, of the three given circles, which is rational in λ. We find in (8) its equation and incidentally prove the theorem.

In conjugate coordinates $z, \bar{z}$, a circle is

$$C_1(z) = a_1 z\bar{z} + \alpha_1 z + \bar{\alpha}_1 \bar{z} + b_1 = 0,$$

where a_1, b_1 are real, and α_1, $\bar{\alpha}_1$ are conjugate imaginary. The bilinear invariant of two circles $C_1(z)$, $C_2(z)$ is

$$[C_1, C_2] = \alpha_1\bar{\alpha}_2 + \alpha_2\bar{\alpha}_1 - a_1 b_2 - a_2 b_1.$$

It vanishes when the two circles are orthogonal. When they coincide it becomes $[C_1 C_1] = 2(\alpha_1\bar{\alpha}_1 - a_1 b_1)$. This vanishes when C_1 is a *point circle,* i.e. one whose equation is

$$(1) \qquad P_{z_i}(z) = (z - z_i)(\bar{z} - \bar{z}_i) = 0.$$

It is easily verified that

$$[C_1, P_{z_i}(z)] = -C_1(z_i); \quad [P_{z_i}(z), P_{z_k}(z)] = -P_{z_i}(z_k) = -P_{z_k}(z_i).$$

The two point circles of the pencil $C(z) + \mu K(z) = 0$ are determined by

$$[C + \mu K, C + \mu K] = [C, C] + 2\mu\, [CK] + \mu^2\, [KK] = 0.$$

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* PHYSIK. ZEITSCHR., vol. 17 (1916), p. 449.

† TRANSACTIONS AMER. MATH. SOCIETY, vol. 22 (1921), p. 101. The geometric proof of this theorem given by Dr. Walsh is very complicated. The method of proof followed here is considered by Dr. Walsh (loc. cit.,

They coincide and the circles $C(z)$, $K(z)$ touch when

$$[KC]^2 - [KK]\,[CC] = 0.$$

We begin the proof with the condition

$$\frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)} = \lambda, \tag{2}$$

and set

$$l_1 = (z_2 - z_3)(z - z_1), \quad l_2 = (z_3 - z_1)(z - z_2), \quad l_3 = (z_1 - z_2)(z - z_3),$$

$$l_1 + l_2 + l_3 = 0; \tag{3}$$

$$q_1 = \lambda(\lambda - 1), \qquad q_2 = \lambda, \qquad q_3 = 1 - \lambda,$$

$$q_2 q_3 + q_3 q_1 + q_1 q_2 = 0. \tag{4}$$

The condition (2), or $(\lambda + l_3/l_1) = 0$, when multiplied by its conjugate, $(\lambda + \bar{l}_3/\bar{l}_1)$, is easily reduced by the use of (3) and (4) to the symmetrical form $q_1 l_1 \bar{l}_1 + q_2 l_2 \bar{l}_2 + q_3 l_3 \bar{l}_3 = 0$. Again, in the notation of (1), this condition is

$$q_1 P_{z_2}(z_3) \cdot P_{z_1}(z) + q_2 P_{z_3}(z_1) \cdot P_{z_2}(z) + q_3 P_{z_1}(z_2) \cdot P_{z_3}(z) = 0.$$

For fixed values of λ, z_2, z_3, z this is the equation which determines z_1. If z_1 lies on a circle $C_1(z) = 0$, then

$$K \equiv q_1 P_{z_2}(z_3) \cdot C_1(z) + q_2 C_1(z_3) \cdot P_{z_2}(z) + q_3 C_1(z_2) \cdot P_{z_3}(z) = 0.$$

For fixed z_2, z_3, and z_1 variable in the circle C_1, $K(z) = 0$ is the equation of the circle within which z lies. Now let z_2 range from its fixed position outward in all directions toward the boundary of a circle C_2. Then the circle $K(z)$ ranges outward in all directions from its original position toward the boundary of an envelope which is the *outer part* of the envelope of the ring of circles $K(z)$ as z_2 runs over the circumference of the circle C_2. This envelope is the locus of points z for which *K regarded as a circle in the variables z_2, $\bar{z}_2$* touches the given circle C_2 and therefore the equation of the envelope is

$$[KC_2]^2 - [KK][C_2C_2] = 0.$$

We shall show that $[KK]$ is a perfect square and therefore the envelope factors into a pair of circles of which we want the outer. We notice that $K(z_2)$ breaks up into three terms, $K_1(z_2) + K_2(z_2) + K_3(z_2)$. Hence

$$[KK] = \Sigma[K_iK_i] + 2\Sigma[K_iK_j] \quad (i, j = 1, 2, 3;\ i \neq j).$$

footnote, p. 102) but rejected because of algebraic difficulties. These however are not inherent. The algebraic method has, moreover, the decided advantage of furnishing the required envelope in covariant form.

To within the coefficients q the terms $[K_iK_j]$ are all alike and equal to $-P_{z_3}(z)\cdot C_1(z)\cdot C_1(z_3)$. But according to (4) the sum of the coefficients of these three terms vanishes. Moreover $[K_iK_i]=0$ $(i=1, 2)$ since $K_1(z_2)$ and $K_2(z_2)$ are point circles. Hence $[KK]=q_3^2\cdot(P_z^3(z))^2\cdot[C_1C_1]$. Also

$$[KC_2] = -q_1C_1(z)\cdot C_2(z_3) - q_2C_1(z_3)\cdot C_2(z) + q_3P_{z_3}(z)\cdot[C_1C_2].$$

Thus for proper choice of the sign of the radicals the outer part of the envelope is the circle

$$(5)\quad L \equiv -q_1C_1(z)\cdot C_2(z_3) - q_2C_1(z_3)\cdot C_2(z) + q_3P_{z_3}(z)\{[C_1C_2] - \sqrt{[C_1C_1]}\sqrt{[C_2C_2]}\}.$$

We now let z_3 run over a circle $C_3(z)=0$. As before the envelope of the circle $L(z)$ in (5) is the tact-invariant of L regarded as a circle in the variable z_3 and of $C_3(z_3)=0$. It is therefore

$$(6)\qquad [LC_3]^2 - [LL][C_3C_3] = 0.$$

Again the term $[LL]$ is a perfect square. In fact

$$[LL] = q_1^2C_1^2(z)\cdot[C_2C_2] + q_2^2C_2^2(z)\cdot[C_1C_1] + 2q_1q_2C_1(z)\cdot C_2(z)\cdot[C_1C_2] + 2q_3(q_1+q_2)C_1(z)\cdot C_2(z)\{[C_1C_2] - \sqrt{[C_1C_1]}\sqrt{[C_2C_2]}\},$$

the term in q_3^2 dropping out since q_3 is the coefficient of a point circle. Since $q_3(q_1+q_2) = -q_1q_2$ this becomes

$$(7)\qquad [LL] = \{q_1C_1(z)\sqrt{[C_2C_2]} + q_2C_2(z)\sqrt{[C_1C_1]}\}^2.$$

Hence the final envelope (6) factors into two circles (necessarily inner and outer) and the theorem is proved.

In order to obtain the equation of the envelope we note that

$$[LC_3] = -q_1C_1(z)\cdot[C_2C_3] - q_2C_2(z)\cdot[C_1C_3] - q_3C_3(z)\cdot\{[C_1C_2] - \sqrt{[C_1C_1]}\sqrt{[C_2C_2]}\}.$$

This, together with (7), yields the factors of (6), whence

The locus of z referred to in the theorem is, in explicit form,

$$(8)\quad \begin{aligned} &\lambda(\lambda-1)\cdot C_1(z)\{[C_2C_3] - \sqrt{[C_2C_2]}\sqrt{[C_3C_3]}\}\\ &+ \lambda\cdot C_2(z)\{[C_3C_1] - \sqrt{[C_3C_3]}\sqrt{[C_1C_1]}\}\\ &+ (1-\lambda)\cdot C_3(z)\{[C_1C_2] - \sqrt{[C_1C_1]}\sqrt{[C_2C_2]}\} = 0;\end{aligned}$$

where the sign of the radical $\sqrt{[C_iC_i]}$ is to be taken opposite the sign of the quadratic q_i for given λ.

On account of the symmetry and homogeneity of this result the verification of sign can be made for (5) and $a_1 = a_2 = 1$. We have in (5) two circles which determine the pencil

$$(9)\quad \mu\{-q_1C_2(z_3)\cdot C_1(z) - q_2C_1(z_3)\cdot C_2(z) + q_3P_{z_3}(z)\cdot[C_1C_2]\} \\ - \{q_3\sqrt{[C_1C_1]}\sqrt{[C_2C_2]}P_{z_3}(z)\} = 0.$$

Let C_1, C_2 be small circles around the points z_1, z_2 respectively which themselves are not on a line with the point z_3. The pencil (9) contains the point circle $P_{z_3}(z)$ and therefore another point circle P interior to the two point circles (5) since $P_{z_3}(z)$ is exterior to them. Hence in (9) we must have for $\mu = 0$ the point circle z_3; for $\mu = \mu_R$ the radical axis; for $\mu = 1$, the outer circle (5); for $\mu = -1$ the inner circle (5); and for $\mu = c(-1 < c < 0)$, the point circle P. Thus μ_R, the parameter of the radical axis of the pencil (9) must be positive. But

$$\mu_R = q_3\sqrt{[C_1C_1]}\sqrt{[C_2C_2]}/(-q_1C_2(z_3) - q_2C_1(z_3) + q_3[C_1C_2]).$$

If now the circles C_1, C_2 approach the points z_1, z_2 as limits the denominator of μ_R approaches as a limit

$$(10)\qquad -(q_1\alpha^2 + q_2\beta^2 + q_3\gamma^2)$$

where α, β, γ are the lengths of the sides opposite the vertices z_1, z_2, z_3 of the triangle z_1, z_2, z_3. In terms of λ (10) becomes

$$(11)\qquad -\alpha^2\lambda^2 + (\gamma^2 + \alpha^2 - \beta^2)\lambda - \gamma^2.$$

The discriminant of (11) is

$$(\alpha + \beta + \gamma)(-\alpha + \beta + \gamma)(-\beta + \alpha + \gamma)(\gamma - \alpha - \beta)$$

which is negative. Hence (11) is a definite quadratic form evidently negative for sufficiently large λ. Then (10) is negative for all real values of λ and this requires that $q_3\sqrt{[C_1C_1]}\sqrt{[C_2C_2]}$ be negative. Since $q_1q_2q_3 = -\lambda^2(\lambda - 1)^2$ is negative for all real values of λ, the three radicals must take the same signs as, or opposite signs to, the three quadratics q,

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ON SKEW PARABOLAS.

BY DR. MARY F. CURTIS.

The theorem that a real rectifiable skew parabola is a helix, proved in my note in this BULLETIN, November, 1918, for skew parabolas which can be represented in rectangular coordinates by equations of the form:

$$(1)\qquad x_1 = at,\qquad x_2 = bt^2,\qquad x_3 = ct^3,\qquad abc \neq 0,$$

was extended by Professor Hayashi in this BULLETIN, November, 1919, to cover all real skew parabolas, whose equations he reduces to the form