$$(9)\quad \mu\{-q_1C_2(z_3)\cdot C_1(z) - q_2C_1(z_3)\cdot C_2(z) + q_3P_{z_3}(z)\cdot[C_1C_2]\} - \{q_3\sqrt{[C_1C_1]}\sqrt{[C_2C_2]}P_{z_3}(z)\} = 0.$$

Let C_1, C_2 be small circles around the points z_1, z_2 respectively which themselves are not on a line with the point z_3. The pencil (9) contains the point circle $P_{z_3}(z)$ and therefore another point circle P interior to the two point circles (5) since $P_{z_3}(z)$ is exterior to them. Hence in (9) we must have for $\mu = 0$ the point circle z_3; for $\mu = \mu_R$ the radical axis; for $\mu = 1$, the outer circle (5); for $\mu = -1$ the inner circle (5); and for $\mu = c(-1 < c < 0)$, the point circle P. Thus μ_R, the parameter of the radical axis of the pencil (9) must be positive. But

$$\mu_R = q_3\sqrt{[C_1C_1]}\sqrt{[C_2C_2]}/(-q_1C_2(z_3) - q_2C_1(z_3) + q_3[C_1C_2]).$$

If now the circles C_1, C_2 approach the points z_1, z_2 as limits the denominator of μ_R approaches as a limit

$$(10)\qquad -(q_1\alpha^2 + q_2\beta^2 + q_3\gamma^2)$$

where α, β, γ are the lengths of the sides opposite the vertices z_1, z_2, z_3 of the triangle z_1, z_2, z_3. In terms of λ (10) becomes

$$(11)\qquad -\alpha^2\lambda^2 + (\gamma^2 + \alpha^2 - \beta^2)\lambda - \gamma^2.$$

The discriminant of (11) is

$$(\alpha+\beta+\gamma)(-\alpha+\beta+\gamma)(-\beta+\alpha+\gamma)(\gamma-\alpha-\beta)$$

which is negative. Hence (11) is a definite quadratic form evidently negative for sufficiently large λ. Then (10) is negative for all real values of λ and this requires that $q_3\sqrt{[C_1C_1]}\sqrt{[C_2C_2]}$ be negative. Since $q_1q_2q_3 = -\lambda^2(\lambda-1)^2$ is negative for all real values of λ, the three radicals must take the same signs as, or opposite signs to, the three quadratics q,

THE UNIVERSITY OF ILLINOIS.

ON SKEW PARABOLAS.

BY DR. MARY F. CURTIS.

The theorem that a real rectifiable skew parabola is a helix, proved in my note in this BULLETIN, November, 1918, for skew parabolas which can be represented in rectangular coordinates by equations of the form:

$$(1)\qquad x_1 = at,\qquad x_2 = bt^2,\qquad x_3 = ct^3,\qquad abc \neq 0,$$

was extended by Professor Hayashi in this BULLETIN, November, 1919, to cover all real skew parabolas, whose equations he reduces to the form

(2) $x_1 = a_1t^3 + a_3t,\ x_2 = b_1t^3 + b_2t^2,\ x_3 = c_1t^3,\ a_3b_2c_1 \neq 0.$

Professor Loria, in an extract from a letter to Professor D. E. Smith published in this BULLETIN, February, 1921, states that Professor Hayashi's work was unnecessary, in that every (real) skew parabola can be represented in rectangular coordinates by equations of the form (1).

When the coordinates are oblique, (1) does represent every real skew parabola.* Professor Loria's statement that this is equally true when the coordinates are rectangular is, however, at fault. A skew parabola C:

(3) $$x_i = a_it^3 + b_it^2 + c_it + d_i \qquad (i = 1, 2, 3)$$

lies on a unique parabolic cylinder S and meets the ruling R on which the vertices of the orthogonal sections of S lie in a unique finite point P. If $P : t = 0$ is taken as the origin of coordinates, R as the x_3-axis, and the axis of the orthogonal section through P as the x_2-axis, equations (3) become

(4) $x_1 = c_1t,\ x_2 = b_2t^2,\ x_3 = a_3t^3 + b_3t^2 + c_3t,\ a_3b_2c_1 \neq 0,$

and these are as simple equations representing a general skew parabola in rectangular coordinates as can be found.†

It is clear that the osculating plane at $P : t = 0$ is the plane $x_3 = 0$ if and only if $b_3 = c_3 = 0$. Then (4) reduces to (1).

THEOREM. *A real skew parabola C can be represented in rectangular coordinates by equations of the form* (1) *if and only if, at the point in which C meets the line of vertices of the parabolic cylinder on which C lies, the osculating plane of C is orthogonal to the cylinder.*

The representation (4) for the general skew parabola, the theorem and the representation (1) for the particular skew parabolas which the theorem singles out hold in the complex domain, unless the rulings of the cylinder S are minimal, or the parallel planes each of which cuts S in a single ruling are minimal, or both the rulings and the planes are minimal. In such cases simple representations of the skew parabola exist.

WELLESLEY COLLEGE.

* ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN, III, C 2: O. Staude. *Flächen 2. Ordnung und ihre Systeme*, p. 234. ENCYCLOPÉDIE DES SCIENCES MATHÉMATIQUES III, 22 (1914), p. 130.

† O. Staude: *Analytische Geometrie der Kubischen Kegelschnitte* (1913), p. 139. The equations (2) which refer the curve to the tangent, principal normal and binormal at an arbitrary point may, for some purposes, be preferable to the equations (4).