CREMONA TRANSFORMATIONS AND APPLICATIONS TO ALGEBRA, GEOMETRY, AND MODULAR FUNCTIONS *

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1. *Introduction.* Two of the most highly developed fields of modern mathematics are those associated with the projective group and the birational group. We have on the one hand projective geometry with its analytic counterpart in the theory of algebraic forms, and on the other hand algebraic geometry and algebraic functions. Between these two domains there lies the group of Cremona transformations for which as yet no distinctive geometry and no distinctive invariant theory has been formulated. It seems opportune therefore to give this brief résumé of achievement in this field along the somewhat scattered lines in which research has been pursued, to indicate certain problems that await solution, and to point out certain directions in which results of importance may be expected.

Several topics are omitted which perhaps first occur to one's mind when Cremona transformations are mentioned. The most important of these omissions is the quadratic transforformation. This, first introduced analytically by Plücker(1)† in 1829 and by Magnus(2) in 1831 as a reciprocal radii transformation, has been the subject of repeated investigation down to recent papers of Emch(3). It is for the geometer a powerful instrument for simplifying a problem or for extending a theorem.

The group of plane motions underlying euclidean geometry may be enlarged in one direction to the projective group and projective geometry or in another direction to the inversive group and inversive geometry. The quadratic transformations of this latter group were first considered in general by Möbius (4) in 1853 under the name of *Kreisverwandschaften.*

* Presented before the Society at the Symposium held in Chicago, April 14, 1922.

† Such numbers refer to the papers listed at the close of this article.

Ten years later, in 1863, Luigi Cremona[5] first established the theory of the general birational point transformation valid throughout the plane. This was extended to space in 1869–71 by Cayley[6], Cremona[7], and Noether[8].

Before taking up this general theory two famous theorems deserve mention. The first states that the general ternary Cremona transformation is a product of quadratic transformations. This was surmised by Clifford[9] in 1869, and was verified by Cayley[10]. Imperfect proofs given by Noether[11] and Rosanes[12] in 1870 were finally replaced in 1901 by a rigorous proof due to Castelnuovo[13]. Unfortunately this theorem has no analog in space, as I shall point out later. The second theorem, proved by Noether[14] in 1871, asserts that any algebraic curve can be transformed by a Cremona transformation into a curve with isolated multiple points with distinct tangents. The curve as thus simplified is a suitable basis for the algebraic functions which Noether had in mind. A proof more geometric in character was given by Bertini[15] in 1888.

A. Cremona Transformations from Space to Space

2. *Classification of Algebraic Correspondences.* This section is largely introductory. Consider the rational transformation*

$$(1) \qquad \begin{cases} y_0 = f_0(x_0, x_1, x_2)^n, \\ y_1 = f_1(x_0, x_1, x_2)^n, \\ y_2 = f_2(x_0, x_1, x_2)^n, \end{cases}$$

from point x of a plane E_x to point y of a plane E_y, where, first, the three curves f of order n do not belong to a pencil,† and second, the three curves f are not compounded of members of a pencil, as would be the case, for example, if they were pairs of lines on a point. In short the jacobian $J(f_0, f_1, f_2)$ does not vanish identically. Then, as x describes the plane E_x, y describes the plane E_y and *vice versa*. We have thereby

* Throughout this paper, homogeneous coordinates are used.

† I shall use the terms pencil, net, and web for a linear system of ∞^1, ∞^2, ∞^3 things respectively.

merely established a projective correspondence between the net of lines $\eta_0 y_0 + \eta_1 y_1 + \eta_2 y_2 = 0$ in E_y and the proper net of curves $\eta_0 f_0 + \eta_1 f_1 + \eta_2 f_2 = 0$ in the plane E_x, where the term proper implies that $J \neq 0$. To the point y, the base of a line pencil in E_y, there corresponds in E_x the variable base points of the corresponding pencil of curves in E_x.

We distinguish here three cases. First, the case $n = 1$, for which the net in E_x is merely the net of lines whose pencils have a single base point variable with the pencil. This is the familiar *projective transformation*. Second, the case in which $n > 1$ and in which the pencils of the net still have a single base point on E_x variable with the pencil. This is the general *Cremona transformation*. The coördinates of the variable base point can be expressed rationally in terms of those of y, and the transformation is birational throughout the planes. Third, the case in which $n > 1$ and in which the pencils of the net have k base points variable with the pencil. This is a 1 to k correspondence between E_y and E_x, and the coordinates x are irrational k-valued functions of the coordinates y. The birationality is restored by a simple device. As x runs over a curve $g(x) = 0$ in E_x, y runs over a curve $h(y) = 0$ in E_y; and, for each position of y on $h(y)$, in general only one of the k corresponding points x on E_x will lie on the given curve $g(x)$, so that equations (1) together with $g(x) = 0$ establish a birational correspondence between x and y which, however, is limited to the curves g, h. To such transformations, one-to-one over limited regions of the linear spaces in question, the term *birätional transformation* will be confined.

If to (1) we add new equations of the same form, say,

$$(2) \qquad \begin{cases} y_3 = f_3(x_0, x_1, x_2)^n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_d = f_d(x_0, x_1, x_2)^n, \end{cases}$$

thereby enlarging the net of curves f to a linear system of dimension d and expanding the plane E_y to the linear space S_d, then we have a *mapping* of the plane E_x upon a manifold $M_2{}^k$ of dimension 2 and order k in S_d, which in general is birational.

3. *Fundamental Points and Principal Curves.* Reverting to the case of the Cremona transformation (1) for which a single base point x of the pencils of the net is variable, we can show[16] that the fixed base points of the pencils are fixed for the entire net, say at $p_1, p_2, \cdots, p_\rho$ with multiplicities $r_1, r_2, \cdots, r_\rho$ respectively. For these points—the *fundamental points,* or *F-points,* of the transformation—the functions f vanish and y is indeterminate. For simplicity I shall assume that these points are isolated. It is, however, a single linear condition on the net that a curve of the net shall pass through an F-point p of multiplicity r with a given direction. To the pencil of curves with this given direction there corresponds on E_y a pencil of lines on a definite point y. The locus of such points y is the *principal curve,* or *P-curve,* on E_y which corresponds to p on E_x, or rather to the directions about p on E_x. It is rational and of order r, since a line on E_y meets it in as many points as the corresponding curve of the net on E_x has directions at p. The P-curves on E_y can meet only at the F-points on E_y of the inverse transformation, say $q_1, q_2, \cdots, q_\sigma$ of multiplicities $s_1, s_2, \cdots, s_\sigma$ ($\sigma = \rho$[17]) for the curves of the net on E_y (also of order n[17]) which correspond to the lines on E_x. To the directions about an F-point on E_y there corresponds a P-curve on E_x which can meet the general curve of the net only at the base points. It must therefore be a fixed rational constituent of a pencil of the net. Conversely if a pencil of the net has a fixed part, this part must correspond to a single point,—necessarily an F-point,—on E_y. Thus if the net (1) is given on E_x, and its base points of multiplicities r are thereby known, the pencils with fixed parts are easily isolated. If also the projective correspondence between lines on E_y and curves of the net is given, the pencils with fixed parts determine the position of the F-points on E_y and the orders of the fixed parts are the multiplicities of these F-points for the net on E_y. The number of constants required to determine the transformation is $2\rho + 8$. If α_{ij} is the number of times the curve P_j on E_y, which corresponds to p_j on E_x, goes through q_i on E_y, it is the number of directions at q_i which correspond to direc-

tions at p_j, and therefore also the number of times the curve P_i on E_x, which corresponds to q_i, passes through p_j. We shall have occasion to use these numbers soon.

Cremona transformations in space are obtained by setting up a projective correspondence between the web of planes in $S_3(y)$ and a *homoloidal* web of surfaces in $S_3(x)$, i.e., a web such that the nets of surfaces in the web have a single variable intersection x which describes completely the space $S_3(x)$. We now find F-points of three kinds which are exemplified by the cubi-cubic transformation determined by three bilinear forms

$$(3) \qquad (\alpha_i x)(\beta_i y) = 0, \qquad (i = 1, 2, 3).$$

Here for general forms a point x determines three planes which meet in a unique point y unless x happens to be on the sextic curve defined by the vanishing of the (3, 4) matrix of coefficients of y. Then the three planes meet in a line every point y of which is a correspondent of the given point x. We say then that x is an F-point of an F-curve *of the first kind* and that its corresponding line is a P-curve whose locus is a *P-surface of the first kind.* The ∞^2 directions at x lie ∞^1 at a time in the ∞^1 planes on the tangent line to the F-curve at x, and all the directions at x in one of these planes correspond to a single point on the P-curve of x.

However, in the particular case when the forms (3) are

$$(4) \qquad \textstyle\sum_{j=0}^{j=3} a_{ij} x_j y_j = 0, \qquad (i = 1, 2, 3),$$

the transformation just considered degenerates into the familiar type,

$$(5) \qquad y_j = r_j/x_j = \rho r_j x_k x_l x_m, \quad (j, k, l, m = 0, 1, 2, 3).$$

The sextic F-curve has become the six edges of a tetrahedron T_x. The point x on such an edge is still an F-point. The ∞^2 directions about it correspond no longer to points on a P-curve, but rather to directions about a definite point on an edge of a tetrahedron T_y in $S_3(y)$. We say that the six edges are *F-curves of the second kind.* They have no corresponding P-surfaces. Another novelty here is the four *F-points of the second kind,* the four vertices of T_x. To the ∞^2 directions about one of these there correspond the ∞^2 points on a plane of T_y so that to an F-point of the second kind there corresponds a P-surface. Moreover the F-curves of the second kind are a necessary consequence of the existence of the F-points of the second kind, since the cubic surfaces of the web with nodes at the vertices of T_x must necessarily contain the edges of T_x. This space transformation is the immediate extension of the quadratic transformation of the plane.

In four dimensions, a Cremona transformation may have F-surfaces, F-curves, and F-points, each of various kinds, and obviously the possible complications increase with the dimension. However, the transformation of the type (5) preserves its form throughout, being determined in $S_k(x)$, $S_k(y)$ by a set of $k+1$ F-points in either space and a pair of corresponding points, all other F-points of the transformation being a necessary consequence of the existence of the given sets.

To replace the theorem which states that in the plane direct and inverse transformations have the same number of F-points, Pannelli(18) proves that, if in space they have respectively σ, σ' F-points and τ, τ' F-curves of genera ρ_i, ρ_i', then

$$\sigma + \tau - \Sigma_i \rho_i = \sigma' + \tau' - \Sigma_i \rho_i'.$$

From the fact that in space the product of two transformations with respectively F-curves of genera ρ_1, ρ_2 will itself have F-curves of genera equal to both ρ_1 and ρ_2, and the further fact that transformations can be constructed with an F-curve of arbitrarily great genus Miss Hilda Hudson(19) concludes that the general space transformation can not be expressed as a product of a finite number of given types.

4. *Types of Cremona Transformation.* If the Cremona transformation of order n in the plane has F-points of multiplicities $r_1, r_2, \cdots, r_\rho$ arranged in descending order of magnitude, we deduce from the fact that the curves of the net are rational, and that they have but a single variable intersection, the necessary conditions

$$(6) \qquad \begin{cases} r_1^2 + r_2^2 + \cdots + r_\rho^2 = n^2 - 1, \\ r_1 + r_2 + \cdots + r_\rho = 3(n-1), \end{cases}$$

on the positive integers $n, \rho, r_1, \cdots, r_\rho$. In addition to these there are an unlimited number of inequalities of the form

$$(7)\qquad \begin{cases} r_1 + r_2 & \leqq n, \\ r_1 + \quad \cdots + r_5 & \leqq 2n, \\ 2r_1 + r_2 + \cdots + r_7 & \leqq 3n, \\ r_1 + \quad \cdots + r_9 & \leqq 3n, \text{ etc.,} \end{cases}$$

which must be satisfied by these integers lest the curves of the net all contain a factor of order 1, or 2, or 3, etc. For given n, the solutions of this Diophantine system are finite in number and the tabulations of such solutions from Cayley's(20) articles to the more recent articles of Montesano(21) and Larice(22) are arranged in this way. It will appear, I hope, that a more natural classification of these solutions is according to the number ρ of F-points. For $\rho < 9$ the number of solutions is finite. For $\rho = 9$ the number is infinite and I have proved(23) that this infinite number can be arranged in $960 \cdot 3^7 \cdot 2$ classes, such that each class contains an infinite number of solutions depending on the unrestricted variation of eight integers. This is by far the most extensive aggregate of solutions as yet formulated.

The transformation (1) transforms a curve of order μ_0 with multiplicities $\mu_1, \cdots, \mu_\rho$ at the F-points on E_x into a curve of order μ_0' with multiplicities $\mu_1', \cdots, \mu_\rho'$ at the F-points on E_y, where

$$(8)\qquad \begin{cases} \mu_0' = n\mu_0 - r_1\mu_1 - r_2\mu_2 \cdots - r_\rho\mu_\rho, \\ \mu_1' = s_1\mu_0 - \alpha_{11}\mu_1 - \alpha_{12}\mu_2 \cdots - \alpha_{1\rho}\mu_\rho, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \mu_\rho' = s_\rho\mu_0 - \alpha_{\rho 1}\mu_1 - \alpha_{\rho 2}\mu_2 \cdots - \alpha_{\rho\rho}\,\mu_\rho. \end{cases}$$

If also pairs of ordinary points are to be considered we add

$$(9)\qquad \mu_{\rho+1}' = \qquad - (-1)\mu_{\rho+1}, \text{ etc.}$$

The coefficients n, r_j, s_i, α_{ij} of this linear transformation determine the *type*(23) of the Cremona transformation. All types of this kind can be generated by that of the quadratic transformation, namely,

$$(10)\qquad \begin{cases} \mu_0' = 2\mu_0 - \mu_1 - \mu_2 - \mu_3, \\ \mu_1' = \mu_0 \qquad - \mu_2 - \mu_3, \\ \mu_2' = \mu_0 - \mu_1 \qquad - \mu_3, \\ \mu_3' = \mu_0 - \mu_1 - \mu_2, \end{cases}$$

together with permutations of the μ's or μ''s. In this way, *for given* ρ, we obtain a linear group with integral coefficients whose elements furnish the solutions of (6), (7), a group whose general modular theory has yet to be studied.

For spaces of higher dimension, we define a *regular* Cremona transformation to be a product of projectivities, and of a single transformation of the type (5), $y_i' = 1/x_i$, $(i = 0, 1, \cdots, k)$. Such regular transformations have properties entirely analogous to those of the plane. For S_3 they have been studied by S. Kantor[24] and for S_k by myself[23]. The preceding arithmetic discussion is generalized in my paper[23].

B. Cremona Transformations in a Single Space

5. *The Cremona Group and its Subgroups.* In this section the planes E_x and E_y are understood to be superposed. We should perhaps speak rather of the Cremona transformation from point x to point x' of the same plane. The totality of such transformations form a group. This group does not depend on a finite number of parameters since an element with ρ F-points depends upon $2\rho + 8$ constants and ρ may be as large as we please. Nor is it an infinite group in the sense of Lie, since it is not defined by differential equations. Though it contains continuous subgroups its most striking property is the *discontinuity* which appears in the variation of the order n of its elements throughout the range of positive integers. One might expect therefore to find within or associated with it a rich array of discontinuous groups both finite and infinite.

Enriques[25] has shown that any finite continuous subgroup in the plane can be transformed by Cremona transformation into one of three types: (*a*) the 8-parameter collineation group; (*b*) the 6-parameter group of inversions; and (*c*) the *Jonquière groups* J_m, which carry into itself a pencil of rays on a point O, and also a linear system of curves of order m with an $(m - 1)$-fold point at O and common tangents at O; or into a continuous subgroup of one of these three types. Enriques and Fano[26] have made a similar study for space with analogous results. Noether[27] has classified continuous groups of quadratic transformations in S_3 into five total groups with various subgroups. In general one may say that any continuous group of Cremona transformations in S_k can be regarded as a projective group in a space S_d, $d > k$, which has an invariant M_k. For the group has an invariant linear system of dimension d which maps S_k upon an M_k in S_d. Mohrmann[28] has determined these surfaces in S_d for the three types of Enriques.

Apart from the projective subgroup, the most important of the groups of Enriques is the inversive group. If the F-points of a quadratic transformation and its inverse are respectively (p_1, p_2, p_3) and (q_1, q_2, q_3), such that directions at p_i correspond to points on q_jq_k, the group of inversions is made up of those quadratic transformations for which the pairs (p_1, p_2) and (q_1, q_2) coincide at the circular points. The applications of this group in metric geometry and function-theory are well known. It may be extended to space in various ways. If the circular points be replaced by the absolute conic at infinity, the group of quadratic inversions in space is obtained. If the pair of circular points be replaced by three lines in space, a quite different group of cubic transformations appears. Young and Morgan[29] have discussed this group and its extensions with particular reference to the cubic curves bisecant to the three lines. These curves are *one* extension of the circle in the plane; but perhaps a more natural extension is to the cubic surfaces containing the three lines. Such surfaces are given analytically by trilinear binary forms as circles are given by bilinear binary forms in z, $\bar{z}$.

Mention should be made at this point of infinite discontinuous subgroups of the Jonquière type. If in the plane we fix (p_1, q_1) at O, the quadratic transformations generate the group of all Jonquière transformations of any order n with $(n - 1)$-fold F-point (both direct and inverse) at O. This group has as an invariant the line pencil at O. It is of the same general character as the Cremona group itself. If in space we consider the regular transformations generated by cubic transformations with F-tetrahedra $(p_1, \cdots, p_4)$, $(q_1, \cdots, q_4)$ for which (p_1, q_1) coincide at a fixed point O, there is obtained a group in ∞ to 1 isomorphism with the plane Cremona group which I[30] have called the *dilation* of the plane Cremona group. It is, in fact, considered as a group on the net of lines through O, indentical with the planar group. If the pairs (p_1, p_2), (q_1, q_2) coincide at (O_1, O_2), we have a dilation of the planar Jonquière group. Finally if the triads (p_1, p_2, p_3), (q_1, q_2, q_3) coincide at (O_1, O_2, O_3), we have the special case of the group of Young and Morgan for which the three lines form a triangle. For all of these cases the extensions to higher space are more or less mechanical.

6. *Fixed Points and Cyclic Sets.* The transformation in the plane of order n, C_n, has $n + 2$ fixed points defined by the vanishing of the matrix

$$\left\|\begin{matrix} x_0 & x_1 & x_2 \\ f_0 & f_1 & f_2 \end{matrix}\right\|$$

outside the F-points. It may, however, have a curve of fixed points accounting for some or all of the isolated fixed points. The order of $C_n{}^2$ is $m \leqq n^2$. As a rule $m > n$, so that $C_n{}^2$ has more fixed points than C_n. The extra fixed points arise from those pairs which are interchanged by C_n. Similarly, the fixed points of $C_n{}^3$ are either fixed points of C_n or points of cyclic triads of C_n. Proceeding in this way, the number of cyclic k-ads may be obtained, a number which must be modified if at any time a locus of ∞^1 cyclic k'-ads (k' a factor of k) should arise. In 1866 Reye[31] discussed the four fixed points of the quadratic transformation and in 1880 S. Kantor[32] determined its cyclic triads. It must be remarked, however, that the fixed points, and in general the cyclic sets, play a minor rôle in the transformation as compared with the F-points.

If every point of the plane is a member of a cyclic k-ad (or for particular points a member of a cyclic k'-ad, where k' is a factor of k), the transformation is *periodic* of period k. The most interesting case is the *involutory* transformation. Bertini[33] has proved that any involution can be transformed into one of four types: (*a*) the harmonic homology; (*b*) the Jonquière involution of order n; (*c*) the Geiser involution of order 8; and (*d*) the Bertini involution of order 17. An involution interchanges a line and an n-ic curve which meets the line in $2k$ involutory pairs and $n-2k$ points on the curve of fixed points. Here k is the so-called *class* of the involution. Involutions of successive classes have been studied by Bertini(34), Martinetti(35), and Berzolari(36), and the types found by them have been reduced by Morgan(37) to the four types of Bertini. However, the class of an involution is a projective rather than a cremonian invariant number for the involution.

For the greater part of our knowledge of transformations of higher period and of finite Cremona groups, we are indebted to the genius of the great geometer S. Kantor. His work in this field is remarkable both for breadth of view and mastery of complicated detail. I shall not have space to outline his methods, which, as set forth in his crowned memoir(38) and his book(39), have been reviewed by Caporali(40) and by Kantor(41) himself. His results have been corrected in important but not essential particulars by Wiman(42). Kantor has considered also periodic transformations in space(43) and has derived the finite groups of regular transformations in S_3(44).

C. Geometric Applications

7. *Canonical Forms.* The applications of Cremona transformations to geometry are more or less adventitious. Broadly speaking, transformations enter into geometry in one of three ways. First, the geometric properties of a given transformation may be studied and it is very commonly true that particular transformations define geometric figures of great interest. Instances of this are given in § 9. Secondly, the elements of a given geometric problem may be such that some or all of them serve to define a transformation whose known properties illuminate the given problem. Third, a given geometric problem may be transformed into a simpler form by a properly chosen transformation. The last is perhaps the most common use of transformations.

In birational geometry, a particular curve or surface is usually studied after it has been birationally transformed into a canonical or normal form. So also in problems which involve linear systems of curves or of surfaces, it is usually convenient first to transform the given system by a Cremona transformation into the simplest possible form before discussing its properties.

In connection with his discussion of involutions Bertini(33) has shown that a pencil or a *complete* net of rational curves (i.e., a net determined by base points alone) can be transformed into a pencil or net of lines; a pencil of elliptic curves into a pencil of order $3r$ with 9 r-fold points; a net of elliptic curves into a net of cubics on 7 points; and a web of genus 2 with 4 variable intersections into the web of sextics with 8 given nodes. Guccia(45) extends these results to linear systems of any dimension of rational and elliptic curves, and for the latter case finds that all can be transformed into either a linear system of elliptic cubics with $v = 0, 1, \cdots, 7$ simple base points; or into a linear system of quartics with two nodes; or into the pencil of order $3r$ with 9 r-fold points. Martinetti(46) considers linear systems of genus 2, and Yung(47) linear systems of any genus. These investigations are reviewed and confirmed by Ferretti(48), following Castelnuovo's revised treatment of the homoloidal net. S. Kantor(49) developed an equivalence theory for linear systems of rational, elliptic, and hyperelliptic curves on the hypothesis that the base points of the system of like multiplicity are not to be separated by the use of algebraic irrationalities, and thereby naturally obtained a larger number of types.

These canonical forms are very useful in connection with mapping problems. When we set as in (1), (3)

$$y_i = f_i(x_0, x_1, x_2) = f_i'(x_0', x_1', x_2'), \quad (i = 0, 1, \cdots, d),$$

where x, x' are co-points under a Cremona transformation, and thereby map the plane E_x upon a 2-way in S_d by means of the linear system of curves f_i, then the linear system f_i' determines precisely the same map. Evidently the essential peculiarities of the map will be more clearly presented if the mapping system is first reduced to a normal form.

If $d = 3$, these maps are the rational surfaces. By these methods Picard[(50)] shows that the only surfaces whose plane sections are rational are the Steiner quartic and the rational ruled surfaces, and Noether[(51)] determines the types of rational surfaces of order 4. Other writers along this line are Caporali, Segre, Del Pezzo and Castelnuovo. If $d = 2$, the map reduces to a plane E_y, and there is established a (1 to k) correspondence between E_y and E_x. Classifications of such correspondences for simpler values of k have been made by Sharpe and Snyder[(52)] and by others.

8. *Birational Transformations.* We may define the birational group of a curve or surface to be that aggregate of birational transformations which transforms the given curve or surface into the totality of curves or surfaces, respectively, which are of the same *class*, that is, birationally equivalent to the original curve or surface. Under this definition, any Cremona transformation belongs to the birational group. The converse is not true, however. For example, the rational plane sextic curve can be birationally transformed into a line, but its order cannot be reduced by plane Cremona transformation. If, however, a birational transformation under consideration, say from plane curve to plane curve, can be effected by a Cremona transformation, then the existence of the latter will necessarily throw much light on the former. For the behavior of the transformation off the curve conditions its behavior on the curve. This transition from birational to Cremona transformation is usually not unique. Thus given a cubic curve, $(\alpha x)^3 = 0$, the pairs of points x, x' for which $(\alpha x)(\alpha x')\alpha_i = 0$ $(i = 0,\ 1,\ 2)$ are corresponding pairs of a birational transformation of the hessian cubic into itself. These pairs x, x' satisfy a net of bilinear relations, and any one of the ∞^2 pencils of bilinear relations in this net determines an involutory quadratic transformation which effects the given correspondence on the hessian. This system of quadratic

involutions is of great help in studying the properties of the hessian as related to the correspondence upon it. More pretentious examples of this sort in connection with surfaces are found in papers of Snyder[53] and of Snyder and Sharpe[54].

9. *Geometric Figures attached to Cremona Transformations and Groups.* In this and the following section I wish merely to indicate some interesting geometric contacts of such particular Cremona transformations and groups.

The Geiser[55] involution G has copoints (x, x') which make up with 7 given points the 9 base points of a pencil of cubics. The locus of lines on which (x, x') coincide is a general quartic envelope for which the 7 points form an Aronhold system of double points.

The Bertini involution B has copoints (x, x') which are simple base points of a net of sextics with nodes at 8 given points. Its fixed points are the ninth nodes of such sextics. It isolates the 120 tritangent planes of a space sextic of genus 4 on a quadric cone[56].

The Geiser[55] involution in space G' has copoints x, x' which make up with 6 given points the 8 base points of a net of quadrics. Its locus of fixed points is the Weddle quartic surface.

The Kantor[57] involution in space K is defined by the fact that quartic surfaces with nodes at 7 given points and on a point x meet in another point x'. The ten nodes of a Cayley symmetroid quartic surface have the characteristic property that the involution K determined by any seven of the nodes has the other three nodes for fixed points.

The system of cubic curves on 5 points of the plane is unaltered by a Cremona G_{16} with quadratic elements. The plane is mapped by this system upon a 2-way of order 4 in S_4 which admits 16 collineations. Projected from a point not on it this 2-way becomes a quartic surface with a double conic[58]; from a point on it a general cubic surface[59].

The extension of this group to space is a Cremona G_{32} which leaves a Weddle surface unaltered. Quadrics on the six nodes of the Weddle surface map it upon a Kummer quartic surface and the G_{32} becomes the collineation G_{16} of the Kummer surface[60].

The two groups just defined can be generalized to a $G_{2^{k+2}}$ in S_k determined by $k + 3$ points, but the further cases have not yet been studied. It is worth noting that in the cases for which k is odd the group will contain an involution like G' which is symmetrically related to the whole set of $k + 3$ points (see [23], p. 369, (29)).

The cross-ratio group of Moore[61] of order $(k + 3)!$ is determined by a base ($k + 2$ points) in S_k. It permutes the system of rational norm curves on the base with the $k + 2$ systems of lines on each of the base points. The form problem of this group consists in the determination of k independent cross-ratios of a binary $(k + 3)$-ic whose invariants are given.

It is clear from the above instances that Cremona transformations are intimately related to important geometric configurations. However the real utility of such transformations for geometric investigation can be realized only after closer study. (See, e.g., Conner, loc. cit. [57].)

D. Algebraic Applications

10. *Difficulties.* I have remarked in the introduction that there exists as yet for the Cremona group no distinctive geometry and no distinctive invariant theory. Much of the literature at hand deals with the transformations in a descriptive way, i.e., their projective properties are developed. On the other hand, writers continually use such properties as the invariance of the genus or the permanence of linear series on an algebraic curve under Cremona transformation; but these are properties which belong properly to the birational group rather than to its Cremona subgroup. Fano[62] remarks:

> "It may be a question whether there *are* properties of curves and surfaces not invariant under the birational group but yet invariant under the Cremona group. On this point no systematic investigations have yet been made."

As against this view, we observe that any rational curve can be birationally transformed into a straight line, but that not every rational curve can be so reduced by a Cremona transformation. There must in the nature of the case be *some* way of stating this latter possibility. What is desired is a language whose terms are invariants of the Cremona group but of no larger group. In projective geometry we speak of a *curve of the nth order;* in birational geometry of a *curve,* including under the term any one of the entire algebraic class; the corresponding term in cremonian geometry has as yet no well-defined content. Though the number of cremonian words is quite small they are not entirely lacking. Thus the term "linear system of curves" is cremonian, for a Cremona transformation transforms such a system into a similar system, whereas a birational transformation, being defined on only a single curve, cannot affect a system. Again the theorem:—"The jacobian of a net of curves is a covariant of the net"—belongs not merely to projective geometry for which it is commonly proved, but rather to cremonian geometry. For if the net N is transformed by any Cremona transformation into a net N', the jacobian J_N of N, the locus of additional nodes of curves of the net N, is necessarily transformed into the jacobian $J_{N'}'$ of the net N'. The very meagreness of the cremonian vocabulary ensures that every additional term introduced into it will lead to a considerable enrichment of the geometry. I trust that this statement will be confirmed when the term *congruence of point sets under Cremona transformation* is discussed later. One should not overlook, however, the satisfactory state of two important problems in the plane; namely, the determination of reduced types for linear systems and for finite subgroups.

11. *Two Fields that admit an Invariant Theory.* Two things must be present before an invariant theory is possible: (1) a group of operations and (2) a field of objects permuted by the elements of the group. These are the bare essentials. In order that such a theory shall be fruitful it is also necessary

that the group shall be reasonably significant and that the field of objects shall be reasonably uniform so that an object and its transform may have in common a sufficient body of properties to form a reasonable basis of comparison. The Cremona group is significant enough, but it is hard to find fairly permanent qualities in any field of objects. Thus the order and the singularities of a single curve as well as a contact of two curves may all be altered by a properly chosen transformation. To be sure the totality of linear systems of curves has the permanence required but the objects of this field are as vague as they are general. Furthermore it is desirable to have a second field of objects so that the theory may present something of the nature of duality.

By the introduction of the class of sets of n points in the plane, sets P_n^2, congruent to each other under Cremona transformation, we obtain a new field of objects permuted under the Cremona group which is in a sense dual to the field of linear systems. Moreover the objects of this field are so concrete and so easily visualized that some success in the development of their invariantive properties may well be expected. The definition of this class is as follows: The planar set P_n^2 of points $(p_1, p_2, \cdots, p_n)$ is congruent to the planar set Q_n^2 of points $(q_1, q_2, \cdots, q_n)$ if there exists a Cremona transformation C_m with $\rho \leqq n$ F-points in the set P_n^2 and ρ inverse F-points in the set Q_n^2 for which the remaining $n - \rho$ points in either set form $n - \rho$ copairs of the transformation C_m.

The invariance of this class under Cremona transformation is an immediate consequence of the definition. For if P_n^2 is congruent to Q_n^2 under C_m and Q_n^2 is congruent to R_n^2 under $C_{m'}'$, then P_n^2 is congruent to R_n^2 under the product $C_m \cdot C_{m'}'$. Nevertheless, certain points relating to the definition should be emphasized. We note first that, P_n^2 being given, not all Cremona transformations are eligible for the formation of the class of congruent sets, but only those with not more than n F-points, and of these only those whose F-points are found in the set P_n^2. This is, to be sure, a limitation, but in applying the operations of any group, we are entitled to prescribe any rules which do not contradict properties inherent in the elements. Moreover the limitation is not serious, for n may be chosen in advance sufficiently large to include any desired point set or any desired transformation.

Again we observe that if, in particular, C_m is a projectivity C_1, the projectively equivalent sets P_n^2, Q_n^2 satisfy the definition of congruence. This of course must be expected since the projective group is a subgroup of the Cremona group. It raises the question, however, as to whether for other Cremona transformations, congruence may not mean merely projectivity. I have shown ([23], p. 353) that this can occur in only four cases, namely, for sets of 5 points under C_2 and C_3; for sets of 7 points under the Geiser C_8; and for sets of 8 points under the Bertini C_{17}. For each of these cases, the introduction of a single additional copair destroys the projectivity.

Finally we observe that congruent point sets are not birationally equivalent. Consider for example an elliptic cubic K^3 on a set P_6^2. A quadratic transformation, A_{123}, with F-points (p_1, p_2, p_3), (q_1, q_2, q_3) and copairs (p_i, q_i), $(i > 3)$, transforms it into an elliptic cubic $K^{3'}$ on the congruent set Q_6^2. In the birational transformation thus effected from K^3 to $K^{3'}$, the set P_6^2 is not equivalent to the congruent set Q_6^2 but rather to that set on $K^{3'}$ which consists of the three points where the sides of the F-triangle (q_1, q_2, q_3) meet $K^{3'}$ again and of the three ordinary points (q_4, q_5, q_6).

Hence in the congruence of point sets we have a notion peculiar to cremonian geometry. It is clear also that the conditions for congruence imply when $n = \rho$ the conditions on the two sets of F-points that the transformation may exist, and when $n = \rho + 1$ the further conditions for the construction of the transformation ([23], § 2). This notion originated in my own mind through the contrast presented by the two algebraic problems of the next section.

12. *Form Problems of Cremona Groups.* A proper conic with three distinct points marked on it is projectively equivalent

to any proper conic with three distinct points on it. We select then such a conic N, and establish on it a parameter system t such that three given points on N have parameters $t = 0, 1, \infty$. Binary quintics, $(\alpha t)^5 = 0$, then determine on N sets of five points. We project four points of such a set into the vertices of the reference triangle and the unit point, and the fifth point takes the position x (x_0, x_1, x_2). Corresponding to the various orders in which this projection can be made, we get 120 points x, a conjugate set under Moore's G_{120}. Fundamental regions for this group have been given by Slaught(63). Thus an ordered binary quintic determines a point x and conversely a point x determines a binary quintic to within projective modification. An invariant of the quintic is symmetric in the roots. Hence the locus of points x for which an invariant of the quintic vanishes is necessarily an invariant curve of G_{120}. The converse of this is also true. The quintic has invariants I_4, I_8, I_{12} of the degrees indicated, and to these there correspond invariant curves of G_{120}, J_6, J_{12}, J_{18} of the orders indicated with multiple points of orders 2, 4, 6, respectively, at the four base points. If the values of the absolute invariants I_8/I_4^2, I_{12}/I_4^3 of the quintic are given, the point x must be on the curves $J_{12}/J_6^2 = I_8/I_4^2$, $J_{18}/J_6^3 = I_{12}/I_4^3$. These meet in $12 \times 18 - 4 \times 4 \times 6 = 120$ points x outside the base points, a conjugate set under G_{120}. The form-problem of G_{120} consists in the determination of one of these points x when the values of the absolute invariants are given. One point is sufficient, since the others are obtained from one by effecting the known operations of G_{120}. If this one point is obtained, the given quintic can be solved by rational processes. For x determines the roots t' of a binary quintic which is projective to the given quintic with roots t. The determination of a linear transformation which sends the known quintic into the required quintic is a long-known exercise in binary forms. Thus the solution of the quintic is reduced to the solution of the form-problem of G_{120}. I shall now show that the form-problem of G_{120} can be solved in terms of Klein's(64) problem of the A's.

To invariants of the quintic there correspond invariant curves of G_{120}; but to irrational invariants of the quintic there correspond members of a linear system of curves invariant under G_{120}. For example, to the irrational invariant $\overline{12}\cdot\overline{23}\cdot\overline{34}\cdot\overline{45}\cdot\overline{51}$ ($\overline{ij} = t_i - t_j$) of weight 5 and degree 2 in each root there corresponds a cubic curve on the four base points. The cubics determined by the 12 conjugates of this irrational invariant all lie in the linear system $\lambda_1 D_1 + \cdots + \lambda_6 D_6$ of cubics on the four base points, the simplest linear system unaltered by G_{120}. Adjoin now the square root of the discriminant Δ of the quintic, an invariant of degree 8. The group reduces to the even G_{60}, the invariants to I_4, $\sqrt{\Delta}$, I_{12}, the invariant curve which corresponds to $\sqrt{\Delta}$ being the six lines on the four base points. Also the above linear system of cubics separates into two nets

$$\lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2, \qquad \mu_0 B_0 + \mu_1 B_1 + \mu_2 B_2,$$

each net being invariant under the Cremona G_{60}. Indeed under G_{60} the members of the first net experience the linear transformations of Klein's group of the A's, a ternary linear g_{60}; and the second net, Klein's group of the B's which arises from that of the A's by replacing ϵ by ϵ^2 ($\epsilon = e^{2\pi i/5}$). The invariants K_2, K_6, K_{10} of g_{60} of orders 2, 6, 10 must be calculated in terms of the known invariants of the quintic, and the same must be done for three invariant forms L_3, L_5, L_7, linear in the B's, and of the degrees indicated in the A's. The form-problem of the A's consists of the determination of A_0, A_1, A_2 when the numerical values of K_2, K_6, K_{10} are given. When this has been done, we find from the known values of the forms L_3, L_5, L_7 and the A_0, A_1, A_2 the values of B_0, B_1, B_2. Knowing A_0, A_1, A_2 and B_0, B_1, B_2, the coordinates of x are easily obtained. Thus the form-problem of the Cremona G_{120} is solved in terms of that of the A's. I have given[(65)] a solution of the A-problem in terms of the binary ikosahedral form-problem, and have carried out explicitly all the calculations necessary for this as well as for the steps outlined above. Finally the binary form-problem is solved in terms of elliptic modular functions. (See Klein, loc. cit. [(63)], p. 132.) The chief rôle of the cross-ratio group in this procedure for the solution of the quintic is to furnish by all odds the most natural reduction to the problem of the A's. It eliminates at one stroke three parameters from the five involved in the given quintic.

I have also discussed(66) the connection of the cross-ratio group with the solution of the sextic and found that it correlated well with the current theory.

These cross-ratio groups are associated respectively with 5 points in the plane, 6 points in space, etc. It is of course a natural question as to whether there is an analogous theory for further sets of points. The case of 6 points in the plane is next in order. When we transform as before four of the six to the reference triangle and unit point, the other two become x_0, x_1, x_2 and y_0, y_1, y_2. In each of these there is an unessential factor of proportionality. One of these factors is removed by requiring that the last coordinate of each shall be the same, leaving one factor of proportionality common to the two points. Thus $P_6{}^2$ has acquired the *canonical form*

$$\begin{matrix} 1, 0, 0 & 1,\ 1,\ 1 \\ 0, 1, 0 & x_0, x_1, u \\ 0, 0, 1 & y_0, y_1, u \end{matrix} \quad .$$

It is represented by a point P in a linear space Σ_4 with coordinates x_0, x_1, y_0, y_1, u. Obviously all projectively equivalent sets $P_6{}^2$ are represented by the same point P in Σ_4; and, conversely, any point P in Σ_4 represents a class of projectively equivalent sets $P_6{}^2$. The ratios of the coordinates of P are absolute projective invariants of the set. If, however, a given set $P_6{}^2$ is transformed into the base points, and a fifth and sixth point in a different order, a new representative point P' in Σ_4 is obtained. The coordinates of P' are rational in those of P, and thereby for all possible permutations of $P_6{}^2$ we find in Σ_4 a set of 6! points P conjugate to each other under a Cremona $G_{6!}$. This is the extension to the larger set of the Moore cross-ratio group. It puts into evidence the projective invariants of the set. These when symmetric are the invariants of the group. I have developed(67) the theory of this group for the set $P_n{}^k$ of n points in S_k and have given a complete system of invariants for P_6^2. Complete systems for further sets would be very useful but they are hard to find.

If to P_6^2 in canonical form we add a point z_0, z_1, u to form P_7^2, the same canonical coordinates appear in the set P_7^3 in S_3, namely:

$$\begin{array}{llll} x_0, y_0, z_0, u & & 0, 0, 0, 1 & \\ x_1, y_1, z_1, u & & 0, 0, 1, 0 & . \\ 1, \ 1, \ 1, \ 1 & & 0, 1, 0, 0 & \\ & & 1, 0, 0, 0 & \end{array}$$

These two ordered sets, the one is S_2 and the other in S_3, are called *associated.* In the general case a set P_n^k is associated with a set P_n^{n-k-2}. Such associated sets are in remarkably intimate relation both in a geometric and algebraic way. For example, they have the same invariants under $G_{n!}$.

If P_6^2 yielded no more than this $G_{6!}$ it would have no advantage over P_6^3 with its cross-ratio $G_{6!}$. Let us see then whether we cannot obtain a little more from P_6^2. To the geometer 6 points in the plane always suggest the cubic surface Γ^3 mapped from the plane by cubic curves on the six points. The directions about each of the 6 points map into 6 skew lines—a line-six—on the surface. The six conics on 5 of the 6 points map into a similar line-six, cutting the first line-six five lines at a time. The two line-sixes form a double-six. The 15 lines on two of the six points map into the remaining 15 lines of the surface. In other words, P_6^2 defines a cubic surface with an isolated line-six. It is known, however, that there are 72 such line-sixes on the surface. They arise as follows. The quadratic transformation A_{123} sends cubics C_i on P_6^2 into cubics C_i' on Q_6^2. The mapping equations are

$$y_i = C_i(x) = C_i'(x'), \qquad (i = 0, 1, 2, 3).$$

Thus the point y on the surface determined by x or x' does not change but the line-six determined by the points of Q_6^2 is that determined at P_6^2 by the sides of the triangle p_1, p_2, p_3 and the points p_4, p_5, p_6. Hence the passage from P_6^2 to the congruent set Q_6^2 corresponds on the surface to the passage from one line-six to another. Thus there must be 72 projectively distinct sets P_6^2 congruent under ternary Cremona transformation to a given set, and these in all possible orders correspond in Σ_4 to $72 \times 6! = 51{,}840$ points P conjugate under a Cremona G_{51840} isomorphic with the group of the lines on a cubic surface and furthermore in immediate algebraic relation to the surface itself. I have called this the *extended group* $G_{6,2}$ of P_6^2 ([23], §§ 3, 4). Its invariants are the invariants of the cubic surface. Its form-problem can be solved in terms of that of Burkhardt's linear group of the Y's which in turn can be solved in terms of hyperelliptic modular functions of genus 2 and rank 3. The group $G_{6,2}$ plays the same rôle in the determination of the 27 lines on a cubic surface as does the cross-ratio group in the determination of the roots of a quintic[68]. Indeed these two problems, which from the Galois point of view present at most a superficial resemblance, can be developed in a series of practically identical stages both in the algebraic and modular function fields. (See [68], p. 372.) This analogy is so perfect as to justify in itself the methods here employed.

The extended group, based as it is on the canonical form of the point set, and on congruence of point sets, can be generalized immediately to the set P_n^k. The associated sets P_n^k and P_n^{n-k-2} have the same extended group. The study of the structure of these extended groups is much simplified by the fact that they are isomorphic to the corresponding arithmetic group of section 4, a linear group $g_{n,k}$ ([63], § 5) with integral coefficients, and eventually in a smaller number of variables than $G_{n,k}$. Except for the sets P_6^2, P_7^2, P_8^2, P_7^3, P_8^4, the number of congruent sets is infinite, and $G_{n,k}$ is infinite and discontinuous. The theory then passes from the algebraic field into that of automorphic functions.

13. *Invariants of a Set P_n^k.* We define an invariant of a set P_n^k under regular Cremona transformation to be an algebraic form in the coordinates of each point of the set which is symmetric in the n points and which is reproduced to within a factor when formed for a congruent set Q_n^k. It is therefore an invariant under $G_{n,k}$. Since projective sets are

congruent also, the invariant must be a sum of terms each of which is a product of determinants formed from the coordinates. We say that the product $\overline{123}\cdot\overline{456}$ formed for P_6^2 is an irrational invariant, since each point of P_6^2 occurs linearly but the symmetry is not present. There are 10 such products and they lie in a linear system of irrational invariants of dimension 4, i.e., all can be expressed linearly with numerical coefficients in terms of 5 of them. In $\Sigma_1^{10}\,(\overline{123}\cdot\overline{456})^2$ we have an invariant of P_6^2 under $G_{6!}$ but not under the extended group $G_{6,2}$. (See [67], §§ 3, 4, 5.)

To make this extension consider the meaning of the determinant factors $\overline{ijk}$ in the products from which invariants are formed. The factor $\overline{ijk}$ vanishes when p_i, p_j, p_k are on a line. This means that the cubic surface mapped from P_6^2 has a node. But it also has a node when two of the points coincide in a definite direction, or when the six points are on a conic. Under the quadratic transformation A_{123}, the condition $\overline{124} = 0$ on P_6^2 becomes for Q_6^2 the condition that q_3, q_4 coincide, and $\overline{456} = 0$ becomes the condition that the six points q are on a conic. We have then 36 discriminant conditions for the set P_6^2, namely, 15 of the type δ_{ij}, which vanishes if p_i, p_j coincide; 20 of the type δ_{ijk}, which vanishes if p_i, p_j, p_k are on a line; and one of the type δ, which vanishes if all six are on a conic. These discriminant conditions are permuted under $G_{6,2}$ like the 36 double-sixes on the cubic surface. Consider now the product $d_2\,\overline{123}\cdot\overline{456}$, where $d_2 = 0$ is the condition of degree two in each point that the six are on a conic. The product is of degree 3 in each point and vanishes once for each coincidence but once more for the coincidences δ_{12}, δ_{13}, δ_{23}, δ_{45}, δ_{46}, δ_{56}, whence it represents a product of the 9 discriminant conditions δ, δ_{123}, δ_{456}; δ_{12}, δ_{13}, δ_{23}; δ_{45}, δ_{46}, δ_{56}. Such a product is one of 40 similar products conjugate under G_{51840}.

They correspond on the surface to the 40 so-called *complexes*. By forming such combinations of these products as are invariant under G_{51840} we obtain the invariants of P_6^2 under the extended group $G_{6,2}$. They are in effect the invariants of the cubic surface and their complete system is known. (See [68], §§ 1, 3.)

Similarly the invariants under $G_{7,2}$ of P_7^2 are the invariants of the quartic envelope determined as in § 9 by the seven points. Their complete system has not been found, but individual members can be obtained by symmetrizing a system of irrational invariants derived from the type

$$\overline{531}\cdot\overline{461}\cdot\overline{342}\cdot\overline{562}\cdot\overline{547}\cdot\overline{217}\cdot\overline{367}.$$

This type is of degree 3 for each point, vanishes once for each coincidence δ_{ij} and in addition vanishes for the seven discriminant factors $\delta_{531}, \cdots, \delta_{367}$. Here again the 63 discriminant factors for the set P_7^2 are the factors of the discriminant of the quartic envelope.

For larger sets beyond P_8^2 in the plane the number of discriminant conditions is infinite and the symmetrizing process under $G_{n,k}$ can no longer be used. Nevertheless in some of these infinite cases algebraic invariants of $G_{n,k}$ do exist. For example, on P_9^2 there is a unique cubic curve, and the invariants S, T of this cubic are invariants of P_9^2 under regular Cremona transformation. For P_{10}^2 the condition that the 10 points be on a cubic curve is a similar invariant. Other instances are for P_9^3 the condition that the 9 points lie on a nodal quadric, and for P_9^4 the condition that the 9 points lie on a rational quintic curve. These and other examples are all suggested by geometric considerations. On the algebraic side, there is no evidence that the invariants which exist for the subgroup $G_{n!}$ can be so combined as to form invariants of $G_{n,k}$ when $G_{n,k}$ is infinite. If the algebraic invariants fail, automorphic functions of the coordinates may replace them.

E. Applications to Modular Functions

14. *Modular Functions of Genus Three.* Our notion of a *modular function* attached to a general algebraic curve is dependent of course upon our definition of the *moduli* of the curve. These may be defined from the algebraic point of view as a set of $3p-3$ independent projective absolute invariants of the normal curve C^{2p-2} in S_{p-1} upon which the

given curve is mapped by means of its canonical adjoints. They may also be defined from the transcendental point of view as the constants ω_{ij} $(i, j = 1, \cdots, p)$ which occur in the periods of the normalized integrals of the first kind attached to the curve. In the latter case, the $p(p+1)/2$ quantities $\omega_{ij} = \omega_{ji}$ are connected by certain relations when $p \geqq 4$ (thus far known only when $p = 4$) so that only $3p - 3$ are independent. In the first case one would naturally define a modular function to be an algebraic function of the algebraic moduli; in the second case an analytic function of the ω_{ij}. But to avoid the vagueness consequent to too great generality, $M(\omega_{ij})$ is restricted to be first a uniform function of the ω_{ij}, and second to be invariant under such integral transformations of the periods as form a subgroup of finite index under the totality of such transformations. It has thereby under the total group a finite number of conjugate functions and is therefore an algebraic modular function. Similarly only such algebraic moduli and such algebraic functions of these moduli are admitted as are uniform functions of ω_{ij}. Since the number of their conjugates is finite, each must be unaltered by a subgroup of finite index of the type mentioned.

The normal curve of genus 3 is the plane quartic which I take, for the moment, as an envelope of class 4. It has 288 Aronhold sets of 7 double points, sets P_7^2. If such a set be taken in canonical form, its representative point P in Σ_6 has coordinates $x_0, x_1, y_0, y_1, z_0, z_1, u$. The ratios of these 7 coordinates are algebraic moduli of the curve with precisely $7!\,288$ conjugate value systems which are conjugate also under the extended group $G_{7,2}$. All of the numerous relations among the 28 double points of C^4 are consequences of the statement that the 288 Aronhold sets P_7^2 are congruent under Cremona transformation. J. R. Conner[57] has derived some of these transformations from a different point of view.

If C^4, now a point curve, be mapped by cubics on the six contacts of a contact cubic, it becomes a sextic curve in space, the locus of nodes of quadrics of a net on 8 base points and the plane sections of this sextic furnish the contacts of a system of contact cubics. In general a set P_8^3 is congruent to infinitely many projectively distinct sets; but if P_8^3 is the set of 8 base points of a net of quadrics, I have shown ([23], p. 377 (45)) that it is congruent to only 36 such sets, each corresponding to one of the 36 systems of contact cubics. In each set P_8^3 there are 8 sets P_7^3 and thus there arise 8×36 sets P_7^3 which are the associated sets of the 288 Aronhold sets P_7^2. The relations among these 36 sets P_8^3 as well as the conditions on a particular set are all implied by their congruence properties.

The irrational invariant of P_7^2 found in § 13 is an algebraic modular function. It appears ([23], § 8) that it is one of 135 conjugates such that each one of the 135 can be linearly expressed in terms of 15 which themselves are linearly independent. Since these 15 are rational functions of only 6 moduli, they are related further, and it turns out that they satisfy a system of 63 conjugate cubic relations. It may be shown that these linear and cubic relations are sufficient to define the system of irrational invariants as functions of the 6 moduli. I shall now show how to obtain expressions for these 15 modular functions as uniform functions of ω_{ij}. Since we can express the six moduli rationally in terms of the 15 functions, we shall thereby have expressions for the algebraic moduli as uniform functions of the transcendental moduli.

Following a suggestion of Klein, Wirtinger[69] has generalized the Kummer surface defined by the theta functions of genus 2 into a p-way M_p in S_{2^p-1} of order $p!\ 2^{p-1}$ defined by the theta functions of genus p. For $p = 3$, this is a 3-way M_3^{24} of order 24 in S_7. The definition is as follows. The squares of the 64 odd and even thetas for $p = 3$ are even functions of the second order and zero characteristic so that only 8 are linearly independent. Using such a set of 8 as homogeneous coordinates of a point in S_7 then, as the variables $u = u_1, u_2, u_3$ change, the point in S_7 runs over the M_3^{24}. A point of M_3^{24} is thus determined by $\pm\, u$; in particular the 64 half-periods (including the zero half-period) determine 64 four-fold points on M_3^{24}. This M_3^{24} is transformed into itself by a collineation G_{64} whose elements are defined parametrically by $u' = u + \frac{1}{2}P$. The 8 theta functions are connected by a system of 70 quartic relations and Wirtinger(70) has proved that all other relations among them are a consequence of these; in other words, M_3^{24} is the complete intersection of 70 quartic spreads in S_7. But, contrary to the case when $p = 2$, there are here 8 cubic relations on the functions, so that 64 of the 70 quartic relations are merely a cubic relation multiplied by one of the 8 functions. I wish to establish these cubic relations.

It is not difficult to find for coordinates 8 combinations of the theta squares, say X_{ijk} of weights $i, j, k = 0, 1$ with the following properties. On adding one half-period the X_{ijk} with odd first weight i change sign (to within a factor common to all), on adding a second half-period those with odd second weight change sign, and similarly for the third. Thus, there arises a collineation G_8 of changes of sign. On adding another half-period the first weights $i = 0, 1$ interchange; similarly for the second, and for the third. Thus there arises a collineation G_8' of permutations of the X which combines with G_8 to yield G_{64}. Assuming now a general cubic relation, we obtain from it, by using G_8, 8 relations, and combine these to obtain a simple relation with only 15 undetermined coefficients $a_1, \cdots, a_{15}$. To this one we apply G_8', and get the 8 distinct cubic relations. The left members of these appear at once as the 8 first derivatives of a quartic relation with coefficients $a_1, \cdots, a_{15}$. Hence *the Wirtinger M_3^{24} is the manifold of double points on a unique quartic spread F^4 in S_7.* To determine the coefficients $a_1, \cdots, a_{15}$ of F^4 we observe that one of the 63 involutions in G_{64} has two fixed S_3' s, each of which cuts M_3^{24} in 16 points determined by specific quarter-periods, and permuted in the S_3 by a G_{16}. Hence this S_3 cuts F^4 in a Kummer surface whose 16 nodes are determined by the quarter-periods, and whose coefficients are linear in $a_1, \cdots, a_{15}$. But the coefficients of a Kummer surface satisfy a cubic relation(71), and thus we find that $a_1, \cdots, a_{15}$ satisfy 63 cubic relations and therefore are proportional to the 15 linearly independent irrational invariants of P_7^2. Moreover the expressions for the coefficients of the Kummer surface in terms of the coordinates of a node are known[71], and, since here the coordinates of a node are the values of X_{ij} for a definite quarter-period, the values of the coefficients $a_1, \cdots, a_{15}$ as uniform functions of ω_{ij} are determined. In turn the expressions for the irrational invariants of P_7^2 as modular functions are obtained.

We have already noted that when P_8^3 is a set of 8 base points of a net of quadrics, the number of sets congruent to it is not infinite as it is for the general set P_8^3. This phenomenon occurs in other cases. Indeed if $E^4(u)$ is the unique elliptic quartic on P_8^3 with elliptic parameter u such that $u^{(1)} + u^{(2)} + u^{(3)} + u^{(4)} \equiv 0 \pmod{\omega_1, \omega_2}$ is the coplanar condition, then any set P_8^3 for which $u_1 + \cdots + u_8 \equiv \omega/r$ (r an integer) has this property. Also in the plane any set P_9^2 which has a similar sum on the unique elliptic cubic through the points has the property. This includes for $r = 1$ the 9 base points of a pencil of cubics and for $r = 2$ the 9 nodes of an elliptic plane sextic. The same peculiarity appears for the ten nodes of a rational plane sextic, and for the ten nodes of a Cayley symmetroid. The reason is that the theorem which states that congruent sets are not projective is not necessarily valid for special point sets. Indeed the theorem was proved ([23], p. 355 (7)) on the hypothesis that the set P_n^k was special to the extent that its n points were taken on an elliptic norm-curve, but the assumption then was made that their elliptic parameters satisfied no linear relation with integral coefficients, an assumption not fulfilled in the cases mentioned above.

15. *Modular Functions of Genus Four.* Consider first the set P_{10}^2 of nodes $p_1, \cdots, p_{10}$ of the rational sextic. The Bertini involution B with F-points at $p_1, \cdots, p_8$ has fixed points p_9, p_{10} whence P_{10}^2 is congruent to itself under B. Moreover under the transform of B by any quadratic transformation, such as A_{ijk}, P_{10}^2 is congruent to itself. Hence P_{10}^2 is self-congruent under the $\binom{10}{8}$ involutions B with F-points at P_{10}^2 and under the conjugates of these involutions

in $g_{10,2}$. The element of the arithmetic group $g_{10,2}$ which corresponds to B has coefficients

$$\begin{array}{cccccccc} 17 & -6 & -6 & \cdots & -6 & 0 & 0 \\ 6 & -3 & -2 & \cdots & -2 & 0 & 0 \\ 6 & -2 & -3 & \cdots & -2 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 6 & -2 & -2 & \cdots & -3 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}$$

and therefore reduced modulo 2 is congruent to the identity. Any transform of B is similarly reducible to the identity. It may be shown(30) that B and its conjugates generate the subgroup $g_{10,2}^{(2)}$ of elements of $g_{10,2}$ which are congruent to the identity modulo 2. Moreover $g_{10,2}^{(2)}$ is an invariant subgroup of $g_{10,2}$ of index $10!\,2^{13}\cdot 31\cdot 51$, whence this is the number of projectively distinct ordered P_{10}^2's congruent to P_{10}^2; or, disregarding the ordering, the number of projectively distinct types of rational sextics which can be derived from a given one is precisely $2^{13}\cdot 31\cdot 51$. Two years earlier than the date (1919) of this result, Miss Hilda Hudson(72) had proved that if a discriminant condition vanished for the sextic, then the sextic could be transformed into one of only 5 kinds, a fact related to the more precise one above.

The factor group of $g_{10,2}^{(2)}$ under $g_{10,2}$ is isomorphic with that subgroup of the odd and even thetas for $p = 5$ which has an invariant even characteristic (loc. cit. (30), p. 248 (9)). In other words the projectively distinct types of congruent sextics are permuted under Cremona transformation according to a theta modular group of the type mentioned.

The set P_{10}^3 of ten nodes of a symmetroid has properties much like the set P_{10}^2 of nodes of a rational sextic. In this case, however, the symmetroid is unaltered not merely by the $g_{10,3}^{(2)}$ but by other transformations so that only $2^8\cdot 51$ projectively distinct symmetroids can be obtained from a given one by regular Cremona transformation and these types are permuted according to the entire theta modular group for $p = 4$. The dependence of these two configurations of nodes upon theta modular functions thus foreshadowed by the occurrence of these factor groups is confirmed by a result obtained by Schottky(73), who proved that by combining theta modular functions of genus four, the coordinates of a set of ten points are obtained which have a characteristic property of the nodes of the symmetroid. His method is not adequate, however, to solve the inverse problem of finding the curve or curves of genus 4 thus attached to a symmetroid. Moreover the rational sextic and the symmetroid mutually determine each other in the following fashion. The plane rational sextic is *conjugate* to a rational sextic in space, the line sections of the one and the plane sections of the other being apolar binary forms. The symmetroid is the quartic locus of planes which cut the space sextic in *catalectic* binary sextics (i.e., sextics reducible to a sum of three sixth powers), the nodal planes being those which cut the sextic in *cyclic* binary sextics (i.e., reducible to two fifth powers). Thus given the plane sextic there is one projectively definite symmetroid; given the symmetroid there are two plane sextics(74). It is this intimate connection between functions of genus 4 and genus 5 which I am attempting to unravel in articles which are appearing in abstract form in current numbers of the PROCEEDINGS OF THE NATIONAL ACADEMY(75).

I have remarked that if algebraic invariants of $G_{n,k}$ are lacking, automorphic functions may be available. I shall now indicate how perhaps such functions may be constructed for the set P_{10}^2 of nodes of a sextic. Since the sextic is congruent to only a finite number of projectively distinct types, it is unaltered by an infinite discontinuous Cremona group, namely the subgroup $\bar{G}$ of $G_{10,\,2}$ which is isomorphic to the subgroup $g_{10,\,2}^{(2)}$ of $g_{10,\,2}$. On the sextic with binary parameter τ, $\bar{G}$ determines a binary group of the form $\tau' = (a\tau + b)/(c\tau + d)$. Let C_i be the cubic curve in variables x on the 9 nodes other than p_i. Transforming the C_i by A_{jkl} we get either cubics C_i or elliptic quartics with a node at p_r and p_s and simple points at the other points p. Divide such a quartic by its canonical adjoint $\overline{p_r p_s x}$, a rational curve determined by points of P_{10}^2, and the quotient thus obtained is still homogeneous of degree 3 in x. If then we take all the elliptic transforms of a given C by $G_{10,2}$ and divide each by its canonical adjoint, we obtain an infinite sequence of such fractional terms. But these terms must be separated into 527 classes such that the terms in one class arise from one term of the class by those operations of $G_{10,2}$ which are in $\overline{G}$, since it is only for such operations that the set of nodes remains fixed. Divide each of the terms in a class by the product π of the ten quadratics q_i in τ determined by the nodal parameters. Consider now the value of such a term when for x we substitute the parametric coordinates of a point τ on the sextic. A term like C_i has for numerator a product of 9 quadratics q_i divided by π and becomes λ_i/q_i where λ_i is numerical; an elliptic quartic term has for numerator $\pi q_r q_s$, and for denominator $\pi q_r q_s q_{rs}$, where q_{rs} is the pair of further parameters of points where the canonical adjoint cuts the sextic outside the nodes, whence it is λ_{rs}/q_{rs}. In other words the terms all reduce to the inverses of those binary quadratics cut out on the sextic by the F-curves of all the transformations of $G_{10,2}$. For the terms in a particular class we determine the numerical factors λ so that these terms are conjugate under $\overline{G}$ and finally sum the $2j$th powers of the terms in each class. Thus for every value of j we have 527 series which formally are pseudo-invariant under $\overline{G}$ and satisfy for $j \geqq 2$ the convergence criterion of Poincaré. Assuming that not all of these vanish for all values of j, then the ratio of any two for the same j will be automorphic under $\overline{G}$ provided $\overline{G}$ is a discontinuous binary group of the type for which the series of Poincaré converge. On returning to the corresponding series in x, we should then have automorphic functions, homogeneous of degree 0 in x_0, x_1, x_2, which converge for a region of values x which includes at least certain regions on the sextic. I have some information concerning the group $\overline{G}$, but not enough to validate entirely the process outlined above. In passing from one sextic to a congruent sextic, the 527 series mentioned would be permuted like the remaining 527 even theta functions for $p = 5$ under the group which leaves one such function unaltered. Similar developments are possible for the symmetroid. If this procedure is valid, we should have functions of two or three variables automorphic under discontinuous groups of far more complex character than any hitherto considered.

16. *The Arithmetic Group. Dickson Groups.* It must be clear from the last section that the information concerning the Cremona group $G_{n,k}$ obtained from the isomorphic linear group $g_{n,k}$ is of considerable importance. This linear group with integer coefficients will have, for every value of the positive integer α, an invariant subgroup (necessarily of infinite order) which consists of those elements congruent to the identity modulo α. The factor group which in concrete form is merely the elements of $g_{n,k}$ reduced modulo α is necessarily finite. I have made[76] a study of this factor $g_{n,k}{}^{(\alpha)}$ for $\alpha = 2$. It appears that 16 cases are to be distinguished according as $n, k \equiv 0, 1, 2, 3 \pmod 4$. In all of these 16 cases, it turns out that this factor group either is itself a simple group, or contains an invariant abelian subgroup of low order whose factor group is simple. The simple groups thus obtained are of known types. Either they are the total group of the odd and even thetas for some value of p or the simple subgroup of this total group which leaves unaltered either an odd or an even function. In the notation of Dickson's *Linear Groups*, these are respectively the simple groups $A(2m, 2)$, $FH(2m, 2)$, and $SH(2m, 2)$.

The question as to the nature of these factor groups for larger values of α, and in particular for values $\alpha = p^n$ (p a prime), has not been touched. We remark of course that $g_{n,k}$ has an invariant linear form and an invariant quadratic form, so that the factor groups are either the linear groups of Dickson with a quadratic invariant or subgroups of them. In the latter case, it may be possible to obtain new series of groups; and even in the former case we should have some new light on the known group and a new application for it.

Investigations along these lines may advance the general problem of an explicit solution for the totality of types of Cremona transformation with a given number of F-points —a problem which amounts merely to asking for the coefficients r_i, s_j, α_{ik}, m of the general element of the arithmetic group $g_{n,k}$.

One would also naturally expect to find that the simpler transformations congruent to the identity modulo α would have interesting geometric properties comparable perhaps to those of the Bertini involution for $\alpha = 2$. It may well be that in this way other sets of points not less striking than the nodes of a rational sextic will be discovered.

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