THE TWENTY-NINTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The twenty-ninth summer meeting of the Society, consisting of two sessions, was held at the University of Rochester, Rochester, New York, Thursday and Friday, September 7–8, 1922, in conjunction with the meeting of the Mathematical Association of America. Many members of the Society attended the special session of the Association on Thursday morning, held at the Research Laboratory of the Eastman Kodak Company, and members of both organizations were invited to visit the buildings of the Bausch and Lomb Optical Company on Friday afternoon. A much appreciated feature of the meeting was the chamber concert given in Kilbourn Hall of the new Eastman School of Music which is an integral part of the University of Rochester. The joint dinner, preceded by a very enjoyable automobile ride arranged by the alumni of the University, was held on Thursday evening at the Oak Hill Country Club, on the future site of the University. At this dinner Professor R. C. Archibald, President of the Association and Librarian of the Society, gave an interesting talk on his impressions of European conditions.

The attendance at the scientific sessions included the following seventy members of the Society:

Archibald, W. J. Berry, William Betz, Borger, W. G. Bullard, R. W. Burgess, Cairns, W. B. Carver, Clarke, H. E. Cobb, Coble, Comstock, Copeland, Crathorne, Decker, Durfee, Eisenhart, Ettlinger, Finkel, Fischer, W. B. Ford, Gale, Gilman, Gummer, Harrell, E. R. Hedrick, Hildebrandt, Hurwitz, Ingels, Karpinski, Kellogg, Kuhn, W. D. Lambert, Lefschetz, Lennes, Long, Matheson, G. A. Miller, Norman Miller, Mirick, C. N. Moore, F. H. Murray, F. W. Owens, H. B. Owens, Pitcher, Plant, Rasor, F. W. Reed, Reid, Harris Rice, R. G. D. Richardson, Ritt, E. D. Roe, J. R. Roe, Seely, Sharpe, Sherk, W. G. Simon, Slaught, C. E. Smith, Virgil Snyder, Swartzel, Watkeys, J. H. Weaver, Wedderburn, D. E. Whitford, F. B. Williams, W. L. G. Williams, Yeaton, J. W. Young.

The Secretary announced the election of the following persons to membership in the Society:

Dr. Beulah May Armstrong, University of Illinois;
Mr. Santiaga Artiaga, City Engineer, Manila;
Professor Francis Easton Carr, Oberlin College;
Dr. Vishnu D. Gokhale, University of Chicago;
Dr. Charles Jordan, Budapest;
Professor Henri Lebesgue, Collège de France;
Mr. Earle Brenneman Miller, University of Wisconsin;
Mr. Edward Everett Rhodes, Mutual Benefit Life Insurance Company;
Mr. Harvey Alexander Simmons, University of Michigan;
Dr. Ernest Bloomfield Zeisler, University of Chicago.

At the meeting of the Council, twenty-one applications for membership in the Society were received.

The Council announced the appointment of the following committees: Professors Haskins, Fiske and H. S. White on award of the Bôcher Memorial Prize; Professors Curtiss, Dunham Jackson and Mitchell on nomination of officers; Professors Coolidge, Fischer, Kellogg, Roever, Tyler, and the Secretary of the Society, on arrangements for the annual meeting; Professors Fite and Dresden and Mr. Joffe on investment of the Eliakim Hastings Moore Fund.

It was announced that the extra volume of the TRANSACTIONS is being printed by the Eschenbach Company, of Easton, Pa.

Greetings were sent to the newly-formed Mathematical Society of Belgium.

At the meeting of the Society, the reciprocity agreement with the London Mathematical Society was ratified, on recommendation of the Council. The medal presented to the Society by the l'Académie royale des Sciences et Beaux-Arts de Belgique, on the occasion of its one hundred fiftieth anniversary, was exhibited. A resolution was passed thanking the University of Rochester, and in particular the department of mathematics, for its generous hospitality.

At the sessions of the Society Professor A. B. Coble presided, relieved by Professors L. P. Eisenhart, G. A. Miller and Virgil Snyder. The session of Friday morning was especially marked by a paper read, at the request of the programme committee, by Professor C. A. Fischer, on *Functions of lines*.

Titles and abstracts of the papers read at this meeting follow below. The papers of Professors Cole, R. L. Moore

and Kline, Dr. Schmidt, Professor Myller, Dr. Zeldin, Professor Graustein, Miss Carlson, Professor Evans, Dr. Franklin, Dr. Murray, Professor Dodd, and the third paper by Professor Schwatt, were read by title. Dr. Camp's paper was read by Professor C. N. Moore, Mr. Cowling's by Professor Ettlinger, Professor Schwatt's first paper by Professor Snyder, and his second by Professor Ettlinger. Mr. Cowling was introduced by Professor Ettlinger.

1. Professor L. P. Eisenhart: *Condition that a tensor be the curl of a vector.*

This paper will appear in the December number of this BULLETIN.

2. Professor S. Lefschetz: *A new class of topological invariants for two-sided manifolds.*

In a two-sided M_n consider a fundamental system Γ_k^1, Γ_k^2, . . . for the k-cycles, $k \leqq \frac{1}{2}n$, and a similar one Γ_{n-k}^1, Γ_{n-k}^2, . . . for the $(n-k)$-cycles; then denote by $(\Gamma_k^i \Gamma_{n-k}^j)$ the number of intersections of the two cycles when a definite sign is attached to each point of intersection after the manner of Poincaré. The elementary divisors of the matrix

$$||(\Gamma_k^i \Gamma_{n-k}^j)||$$

are the invariants in question. It is readily seen that for an M_2 they are all equal to unity. Remarkably enough this is also true for an algebraic surface (though not for the most general M_4), also for a d-dimensional algebraic variety and $k = 1, 2$. As the topological invariants already known have rather arbitrary values for algebraic surfaces, it would seem that their relations with those just defined are unlikely to be simple.

3. Professors F. R. Sharpe and Virgil Snyder: *The* (1, 2) *quaternary correspondences associated with certain space involutions.*

In this paper Professors Sharpe and Snyder use the complex of lines determined by the pairs of conjugate points of a given space involution to find the (1, 2) quaternary correspondence associated with the involution. The method can be applied

to known involutions that are not readily discussed by the inverse process. The non-monoidal cubic involution is proved to be rational. The special case of the cubic inversion is of particular interest because of the complicated configuration of the fundamental elements.

4. Professor C. N. Moore: *On the summability of the double Fourier's series.*

The principal result of this paper is the theorem that the double Fourier's series corresponding to a function of two variables $f(x, y)$ that is integrable (Lebesgue) in a certain region of the (x, y)-plane is summable $(C1)$ to the value of the function throughout the region, except for a set of points of measure zero. This is an extension to double Fourier's series of a theorem with regard to ordinary Fourier's series due to Lebesgue. A further result of the paper is an extension to double Fourier's series of Fejér's theorem under wider conditions than have hitherto been found.

5. Professor N. J. Lennes: *The theory of sets and the foundation of arithmetic.*

The commutative and associative laws of addition and multiplication as well as the distributive law of multiplication are easily shown to hold under any set of axioms sufficient for the ordinary theory of sets. In the last analysis, integers may be regarded as counters for "things," and hence it is proposed to make the properties of integers under the operations of addition and multiplication depend logically upon the corresponding properties of sets.

This paper gives an outline of the process by which the usual axioms of arithmetic may be derived as theorems from the corresponding propositions in the theory of sets.

6. Professor F. N. Cole: *Kirkman parades.*

This paper will appear in the December number of this BULLETIN.

7. Professors R. L. Moore and J. R. Kline: *On the definition of a simple closed surface.*

The authors suggest the following definition for a simple closed surface: A simple closed surface is a closed, bounded and connected point set M such that (a) every simple closed

curve J which belongs to M is the common boundary, with respect to M, of two mutually separated connected subsets of M whose sum is $M - J$. (*b*) If P is a point of M and ϵ is a positive number, there exists in M a simple closed curve J such that, of the two mutually separated connected sets into which M is divided by the omission of J, one contains P and is of diameter less than ϵ. If in a euclidean space of three dimensions M is a point set satisfying this definition, then the authors prove that there is a one-to-one continuous correspondence between the point set M and the surface of a sphere.

8. Dr. Karl Schmidt: *The theory of functions of one Boolean variable.*

In this paper, the expression $f(x) = ax + b\bar{x}$ of the Boolean algebra, or algebra of logic, is studied by new methods, which lead to new results and throw new light on old results. The principal references are to the works of E. Schröder and Eugen Müller. The paper will be published in the TRANSACTIONS OF THIS SOCIETY.

9. Professor A. Myller: *Representation of rectilinear motion by the geodesics of a surface.*

Professor Myller shows that the world-lines, in the Minkowski sense, of any rectilinear motion are in one-to-one correspondence with the geodesics of a surface of revolution, and conversely.

10. Dr. S. D. Zeldin: *Note on steady fluid motion.*

This paper appeared in full in the July number of this BULLETIN.

11. Professor W. C. Graustein: *Real representations of analytic complex curves.*

In a previous paper, presented to the Society in December, 1919, all the representations of a complex point by pairs of ordered real points which satisfy certain general conditions were determined. In the present paper application is made to analytic complex curves, in the plane and in three-dimensional space, of certain of these representations, namely, those invariant under direct transformations of similarity.

12. Miss Elizabeth Carlson: *Extension of Bernstein's theorem to Sturm-Liouville sums.*

The following theorem is proved in this paper: The maxi-

mum of the absolute value of the derivative of a Sturm-Liouville sum of order n does not exceed nhM, where M is the maximum of the absolute value of the sum itself, and h is a constant independent of n and of the coefficients in the sum. The corresponding theorem for finite trigonometric sums is due to S. Bernstein.

13. Professor G. C. Evans: *A Bohr-Langmuir contact transformation.*

The author considers the relation between the Bohr and Langmuir atoms, the latter having a quantized force, with regard to the question as to whether the "events" of one system may be transformed into the events of the other. It is shown that one cannot be the transform of the other by any contact transformation strictly so-called, which does not involve the time explicitly, but that the Langmuir atom can be obtained from the Bohr atom by a transformation which does not involve the time and is a contact transformation "im kleinen," although not "im grossen." The vibrating Langmuir atom is seen to be equivalent to the Bohr atom with elliptical electron orbit; the completely static atom is equivalent to the Bohr atom with circular orbit.

14. Professor G. C. Evans: *An elementary theory of competition.*

The author discusses postulates for monopoly, cooperation and two different sorts of competition, and analyzes them with reference to a simple economic system. The order of prices is obtained, and the effects of price fixing, of excess profits taxation and of certain typical changes in the cost function.

15. Professor G. A. Miller: *Groups in which the number of operators in a set of conjugates is equal to the order of the commutator subgroup.*

The author first proves that every abelian group can be used as a commutator subgroup of some group which involves a set of conjugate operators whose number is equal to the order of the commutator subgroup. He then determines the properties of the system of groups such that every group of the system contains a set of conjugate operators whose number is equal to the order of the commutator subgroup diminished

by one. In particular, he proves that every group of such a system must also contain a set of conjugate operators whose number is equal to the order of the commutator subgroup, and that this commutator subgroup is always an abelian prime power group of type (1, 1, 1, $\cdots$).

16. Dr. C. C. Camp: *Expansions in terms of solutions of partial differential equations. First paper: Multiple Fourier expansions.*

By assuming u to be a product of functions of the separate variables the author finds a solution of $\Sigma_{i=1}^{p} \partial u/\partial x_i + \lambda u = 0$ for the region $-\pi \leqq x_i \leqq \pi$ and the boundary conditions which make u identical for both ends of the several x_i-intervals. This depends on the solution of p ordinary differential equations with boundary conditions, each containing a different parameter. The expansion of a function f in p variables, which is made up of a finite number of pieces, each real, continuous, and possessing continuous partial derivatives, involves solutions of the adjoint systems and is shown by an extension of Professor Birkhoff's method to converge uniformly to f inside the subregion of any such piece. At a point of discontinuity the series, which is always expressible as a multiple Fourier series, converges to the mean value. By a suitable interpretation this is also true on the boundaries. This is believed to be the first adequate proof of convergence for a multiple Fourier series in more than two variables.

17. Professor E. H. Clarke: *On the minimum of the sum of a definite integral and a function of a point.*

Hadamard, in his *Leçons sur le Calcul des Variations*, has given, in § 159, pp. 176–177, a short notice of this problem. It may be stated briefly as follows: Given two points P_1 and P_2 in the xy plane and two functions $\varphi(x, y)$ and $f(x, y, y')$ of class C_{111}, to find a curve C, joining P_1 to P_2, and a point $\pi(x, y)$, lying on the curve C, such that the sum of the numerical value of the integral of $f(x, y, y')$ taken along C and the numerical value of the function $\varphi(x, y)$ for the point π shall be as small as possible. It is proved in this paper that in addition to the usual necessary conditions of the calculus of variations for the arcs $\rho_1\pi$ and $\pi\rho_2$ there are certain other necessary conditions which must hold at the point π and for the family of broken extremals with corners on a curve K through π. These conditions, when slightly strengthened, are

shown to be sufficient to insure the existence of a minimum for the above sum.

18. Professor H. J. Ettlinger: *A simple proof of a fundamental lemma concerning the limit of a sum.*

The author gives a proof of the following lemma upon which may be based the theory of the Riemann integral.

Let the interval $I : a \leqq x \leqq b$ be divided into n equal sub-intervals, $I(i, n)$, each of length $\Delta_n x = (b - a/n)$. To each $I(i, n)$ there corresponds a number $h(i, n)$. Let $|h(i, n)| \leqq M$ for all values of $i \leqq n$ and n, where M is a constant. If x is any fixed value in I, then for each value of n, x is contained in at least one of the sub-intervals. Let it be designated by $I(x, n)$. For each fixed x and each subdivision $I(x, n)$ let the corresponding number be $h(x, n)$, and let $\lim_{n\to\infty} h(x, n) = 0$. Then $\lim_{n\to\infty} \sum_1^n h(i, n)\Delta_n x = 0$.

The proof makes use of simple limit theorems such as are to be found in every calculus textbook.

19. Mr. A. H. Cowling: *Application of Duhamel's theorem to the convergence proof for approximate solutions of differential equations.*

Runge, Heun and Kutta have given certain formulas for the approximate solution of the differential equation $y' = f(x, y)$. Assuming the conditions for which the Cauchy polygon converges, Mr. Cowling shows that these approximations approach the true solution as a limit. In this proof, use is made of Duhamel's theorem in the form given by Professor Ettlinger (see forthcoming paper in the AMERICAN MATHEMATICAL MONTHLY).

20. Dr. Philip Franklin: *Two theorems on multiple integrals.*

This paper will appear in the December number of this BULLETIN.

21. Professor E. L. Dodd: *An extension of the theorem of Bayes, by the use of a certain limit.*

If an observed event E must have been preceded by one of the mutually exclusive conditions C_i, $i = 1, 2, \cdots$; and $f(i)$ is the *a priori* probability that C_i would obtain; and $p(i)$ is

the "productive" probability that C_i would yield E; then the *a posteriori* probability $P(r)$ that C_r preceded E is $f(r)p(r) \div \Sigma f(i)p(i)$. If, however, $f(x)dx$ is the probability that an error will lie between x and $x + dx$, and all real values are postulated as equally likely *a priori*, then $f(x) \equiv 0$,—if, indeed, it can be defined at all. The following extension is suggested to meet this difficulty:

$$P(r) = \lim_{t \to \infty} \frac{f(r, t)p(r)}{\sum_{i=-t}^{i=t} f(i, t)p(i)}.$$

Integration may replace summation. The author gives an example justifying this extension.

22. Dr. W. L. G. Williams: *Fundamental systems of protomorphic formal modular seminvariants of binary forms.*

The author defines fundamental systems of formal modular seminvariants, modulo p, of one binary form and of several binary forms and finds fundamental systems in both cases. These are remarkable in that each of them contains only one formal modular seminvariant not congruent, modulo p, to an algebraic seminvariant. Sufficient conditions for the existence of certain algebraic invariants are derived from the theory of formal modular concomitants and a connection with Hermite's law of reciprocity is noted.

23. Professor O. D. Kellogg: *An example in potential theory.*

The question of the extreme of generality of which the Dirichlet principle is susceptible is closely connected with the question of finite discontinuities of harmonic functions, where by discontinuity is meant a discontinuity either of the function or of a first derivative. It is well known that a harmonic function of x and y cannot have a finite non-removable discontinuity at an isolated point. Professor Kellogg gives a typical example of a harmonic function having finite non-removable discontinuities at every point of a nowhere dense perfect point set of measure zero. Green's function for the infinite plane, bounded by this set, is also established.

24. Mr. F. H. Murray: *On certain systems of differential equations containing a parameter.*

Systems of differential equations, of the form

$$dx_i/dt = X_i(x_1, \cdots, x_n, v, t), \qquad (i = 1, 2, \cdots, n)$$

where X_i is analytic in v in the neighborhood of v_0 and satisfies the usual conditions with respect to $x_1, \cdots, x_n, t$ in a certain domain $|t - t_0| < a$, $|x_i - x_i^0| < b$, have been discussed by Poincaré, Picard, and others. In this paper the writer proposes to give a similar discussion of the system

$$dx_i/dt = X_i(x_1, \cdots, x_n, w)$$

where X_i satisfies the usual conditions with respect to $x_1, \cdots, x_n$, and is periodic of period 2π with respect to the variable $w = vt$, for large values of v. An application is made to a problem related to the restricted problem of three bodies.

25. Mr. F. H. Murray: *Periodic solutions in the problem of three bodies.*

In *Les Méthodes Nouvelles de la Mécanique Céleste,* Poincaré gave a discussion of the existence of periodic solutions of the third kind in the problem of three bodies, which has formed the basis for later researches by Poincaré and other writers. It is the purpose of this paper to present an alternative demonstration of the existence of these solutions, using certain results of von Zeipel.

26. Professor C. A. Fischer: *Functions of lines.*

The first part of this paper contains an elementary treatment of such concepts as the continuous functional, or, as it is sometimes called, function of a line, that is, curve, the linear functional and the derivative and differential of a functional. Then completely continuous transformations and Stieltjes integral equations are discussed briefly.

27. Professor I. J. Schwatt: *The expansion of a certain function.*

The author shows how the expansion of

$$(a_1 \sin^{p_1} x + a_2 \cos^{p_2} x)^n,$$

in powers of x, where p_1 and p_2 are positive integers and n is any real number, depends on the expansion

$$\sin^{p_1} x \cos^{p_2} x = \frac{1}{2^{p_1+p_2}} \sum_{\kappa=0}^{\infty} (-1)^{\kappa} \frac{x^{2\kappa+p_1}}{(2k+p_1)!} \sum_{\alpha=0}^{p_2} \binom{p_2}{\alpha} \times \sum_{\beta=0}^{p_1} (-1)^{\beta} \binom{p_1}{\beta} (p_1 + p_2 - 2\alpha - 2\beta)^{2\kappa+p_1}.$$

28. Professor I. J. Schwatt: *The summation of a family of deranged series.*

The author finds the sum of the series obtained by retaining throughout

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{a + nh}$$

groups of p successive terms, beginning with the first term, and omitting q successive terms after each of these groups. In this case the known tests fail to disclose the conditions under which the resulting series is convergent or divergent. These conditions are determined from the sum of the series. The series is convergent for all values of p and q, except when p and q are both odd, in which case the series is divergent.

29. Professor I. J. Schwatt: *The sum of the harmonic series.*

The author has developed an expression for $Q_{n,k}$, the sum of the products of 1, 2, 3, $\cdots$, n taken k at a time, which he believes to be new.

$$Q_{n,k} = \prod_{\alpha=1}^{n} \Big(\sum_{\substack{k_\alpha = k+1 \\ \alpha-1}}^{n-k+\alpha} k_\alpha \Big), \qquad k_\alpha = 0,$$

$$= \frac{(-1)^k}{n!} \binom{n}{k} (n + 1 + k)! \sum_{\alpha=1}^{k} \frac{1}{(n + 1 + \alpha)(k + \alpha)!} \binom{k}{\alpha} \sum_{\beta=1}^{\alpha} (-1)^\beta \binom{\alpha}{\beta}_\beta^{k+\alpha};$$

then

$$Q_{n,n-1} = n! \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n-1} \frac{n + k}{k!} \binom{2n}{n - k - 1} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha}_\alpha^{n-1+k}.$$

By means of this result the sum of the harmonic series is expressed in powers of n.

R. G. D. RICHARDSON,
Secretary.