A PROPERTY OF CERTAIN FUNCTIONS WHOSE STURMIAN DEVELOPMENTS DO NOT TERMINATE *

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Let $[u_k(x)]$ be a set of continuous Sturmian functions defined on the interval (a, b), i.e., solutions of an equation of the form

$$\frac{d}{dx}\left[k(x)\frac{d}{dx}u(x)\right] + \lambda g(x)u(x) = 0, \tag{1}$$

each satisfying two linear homogeneous self-adjoint boundary conditions and corresponding to a value of λ for which this is possible. We assume that the coefficients of the differential equation have derivatives of all orders, and that $g(x)$ does not vanish in the closed interval (a, b), nor $k(x)$ on the open interval. Let $f(x)$ denote a function with derivatives of all orders, satisfying boundary conditions to be specified presently. We proceed to call attention to a property which such functions must have if their developments in series of the functions $u_k(x)$ are not to terminate.

We denote by a_k the kth generalized Fourier coefficient of f:

$$a_k = \int f u_k g dx, \tag{2}$$

where we have omitted argument x, and limits of integration, a and b. No ambiguity will result from the abbreviation. If in the integral, we replace $u_k g$ by its value obtained from the differential equation (1), and integrate by parts, we obtain

$$a_k = -\frac{1}{\lambda_k}\int (kf')' u_k dx, \tag{3}$$

where the integrated terms have been omitted on the assumption that f satisfies the same self-adjoint boundary conditions as the u_k. Under this assumption, they vanish. We define a series of functions as follows:

$$f_n = -\frac{(kf'_{n-1})'}{g}, \qquad f_0 = f. \tag{4}$$

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We now subject f to the hypothesis that it, and these derived functions, all satisfy the same boundary conditions as the u_k. It then appears that for all n,

$$a_k = \frac{1}{\lambda_k^n} \int f_n u_k g dx. \tag{5}$$

This is a simple generalization of the long familiar equation for the coefficients of a Fourier series. It is rather in the inference drawn from it, than in the generalization itself, that the interest lies. From the equation (5) we pass to an inequality. Let B_k denote the maximum of $|(b - a)g(x)u_k(x)|$, and F_n the maximum of $|f_n(x)|$. Then, evidently,

$$|a_k| \leqq B_k F_n / |\lambda_k|^n, \qquad \text{or} \qquad F_n \geqq (|a_k|/B_k)|\lambda_k|^n. \tag{6}$$

It follows that *unless the development of $f(x)$ terminates, F_n must, for all n greater than a determinable number N, exceed any exponential function of n, Ae^{pn}*, because of the known property of the characteristic numbers λ_k of having infinity as the only limit point of their absolute values.

In other words, *if F_n is less than any such exponential function, for positive A and p and for infinitely many values of n, $f(x)$ is a homogeneous linear function of the $u_k(x)$ with constant coefficients.*

This property takes on special interest when k and g are constant, for in this case the f_n are proportional to derivatives of f. With suitable boundary conditions, it then takes the form: *if the periodic function $f(x)$ has derivatives of all orders, it is either a trigonometric polynomial, or else the maximum of the absolute value of its nth derivative exceeds any exponential function Ae^{pn} for all n from a certain one on.**

In the case of analytic functions of a complex variable, we have the result:

Either $f(z)$ is a polynomial, or else the maximum of the absolute value of its nth derivative on any circle lying entirely within its domain of analyticity exceeds any exponential function Ae^{pn} from a given n on.

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* The result in this particular form was announced to the Society Dec. 2, 1911. See this BULLETIN, vol. 18, p. 234.